

## OPERATOR-WEIGHTED COMPOSITION OPERATORS ON VECTOR-VALUED BLOCH SPACES

MOSTAFA HASSANLOU

(Communicated by I. M. Spitkovsky)

*Abstract.* Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi$  be an analytic operator-valued function on  $\mathbb{D}$ . Then the operator-weighted composition operator  $W_{\psi,\varphi}$  is defined by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z)), \quad z \in \mathbb{D},$$

where  $f$  is an analytic function  $\mathbb{D} \rightarrow X$ ,  $X$  is any complex Banach space. In this paper by considering  $W_{\psi,\varphi}$  on vector-valued Bloch spaces, some qualitative properties of these operators will be characterized.

### 1. Introduction

Let  $\mathbb{D}$  be the open unit disc in the complex plane  $\mathbb{C}$  and, for any complex Banach space  $X$ ,  $H(\mathbb{D}, X)$  denotes the space of all analytic functions  $f: \mathbb{D} \rightarrow X$ . Recall a vector-valued function  $f: \mathbb{D} \rightarrow X$  is analytic if for every  $x^* \in X^*$  the function  $x^* \circ f: \mathbb{D} \rightarrow \mathbb{C}$  is analytic in the classical sense. Let  $L(X, Y)$  be the space of bounded linear operators  $X \rightarrow Y$  where  $X$  and  $Y$  are complex Banach spaces. Let  $\psi: \mathbb{D} \rightarrow L(X, Y)$  be an analytic function and  $\varphi$  an analytic self-map of  $\mathbb{D}$ . Then we define the operator-weighted composition operator  $W_{\psi,\varphi}$  by  $f \mapsto \psi(f \circ \varphi)$ , that is

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z)), \quad z \in \mathbb{D}, \tag{1.1}$$

for  $f \in H(\mathbb{D}, X)$ .  $W_{\psi,\varphi}$  is a linear map  $H(\mathbb{D}, X) \rightarrow H(\mathbb{D}, Y)$ . We have  $W_{\psi,\varphi} = M_{\psi}C_{\varphi}$ , where  $M_{\psi}$  is the operator-valued multiplier  $f \mapsto \psi f$  and  $C_{\varphi}$  is the composition operator  $f \mapsto f \circ \varphi$ . Thus the operator-weighted composition operators are a large class of operators contains other classes in the vector-valued or scalar-valued setting. For example, if  $X = Y = \mathbb{C}$  then we have weighted composition operators  $W_{\psi,\varphi}f(z) = \psi(z)f(\varphi(z))$ , which are the generalization of multiplication operators  $M_{\psi}f(z) = \psi(z)f(z)$  and composition operators  $C_{\varphi}f(z) = f(\varphi(z))$ . These operators have been studied extensively on several analytic Banach spaces. In the vector-valued setting, weighted composition operators have been studied widely on vector-valued Hardy, Bergman, Dirichlet, Bloch and BMOA spaces, see [2, 6, 8, 9, 10, 11, 12, 13, 18]. Weighted compositions appear naturally: for a large class of Banach spaces  $X$ , all

*Mathematics subject classification* (2010): 47B33, 46E15, 46E40, 47B07.

*Keywords and phrases:* Analytic functions, Banach space, compactness, composition operator, weak compactness, vector-valued function.

linear onto isometries between  $X$ -valued  $H^\infty$  spaces are of the form (1.1) for suitable  $\psi$  and  $\varphi$ .

Operator-weighted composition operators are a new subject in the study of operators on analytic function spaces and are defined in [14] and [11]. Laitila and Tylli [11] characterized boundedness and (weak) compactness of these operators on  $H_v^\infty(X)$  spaces and the author(s) in [14] and [3] obtained the same results for locally convex spaces of analytic vector-valued functions.

In this paper we are going to investigate operator-weighted composition operators on vector-valued Bloch spaces. Boundedness and (weak) compactness of these operators will be characterized.

Let  $X$  be a complex Banach space and  $\alpha > 0$ . The vector-valued Bloch space  $\mathcal{B}^\alpha(X)$  is the set of all analytic functions  $f : \mathbb{D} \rightarrow X$  such that

$$\|f\|_{\mathcal{B}^\alpha(X)} = \|f(0)\|_X + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \|f'(z)\|_X < \infty.$$

Also the little vector-valued Bloch space  $\mathcal{B}_0^\alpha(X)$  is the closed subspace of  $\mathcal{B}^\alpha(X)$  consisting of the analytic functions  $f$  with the property  $\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha \|f'(z)\|_X = 0$ . If  $\alpha = 1$ , we simply write  $\mathcal{B}(X)$  and  $\mathcal{B}_0(X)$ . In the scalar-valued case  $X = \mathbb{C}$ ,  $\mathcal{B}^\alpha(X) = \mathcal{B}^\alpha$ . It is proved that (see [15])

$$\|f\|_{\mathcal{B}} \approx \sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{B_p},$$

where  $\varphi_a$  is the Mobius transformation  $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$  and  $B_p$  is the Bergman space consisting of all analytic functions  $f : \mathbb{D} \rightarrow \mathbb{C}$  for which

$$\|f\|_{B_p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty.$$

Here  $dA(z)$  is the normalized area measure on  $\mathbb{D}$  and  $p > 0$ . Moreover, an analytic function  $f : \mathbb{D} \rightarrow X$  is in the vector-valued Bergman space if

$$\|f\|_{B_p(X)}^p = \int_{\mathbb{D}} \|f(z)\|_X^p dA(z) < \infty.$$

The organization of the paper is as follows. In section 2, a norm estimate of the operator is obtained. Section 3 is related to the compactness and weak compactness of the operator-weighted composition operator. Finally, (weak) compactness of the operator  $T_\psi$  which is essential for the study of  $W_{\psi, \varphi}$  will be discussed. The operator  $T_\psi : X \rightarrow \mathcal{B}(Y)$  is defined by  $x \mapsto \psi(\cdot)x$  which is a new ingredient in the vector-valued context. We also present some examples.

Throughout the paper all constants are denoted by  $c$  which may vary from one position to another. For two values  $A$  and  $B$ ,  $A \approx B$  means that there are positive constants  $c_1$  and  $c_2$  such that  $c_1 B \leq A \leq c_2 B$ . Also  $A \lesssim B$  means that there exists a positive constant  $c$  such that  $A \leq cB$ .

### 2. Boundedness

In this section we find norm estimate of  $W_{\psi,\varphi} : \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$ . First we recall the following lemma which is essential for our proof.

LEMMA 1. [19, Lemma 2.1] *For  $\alpha > 0$  and any complex Banach space  $X$ , if  $f \in \mathcal{B}^\alpha(X)$ , then*

1.  $\|f(z)\|_X \leq c\|f\|_{\mathcal{B}^\alpha(X)}$  for any  $z \in \mathbb{D}$  and  $0 < \alpha < 1$ ;
2.  $\|f(z)\|_X \leq c \log \frac{2}{1-|z|^2} \|f\|_{\mathcal{B}^\alpha(X)}$  for any  $z \in \mathbb{D}$  and  $\alpha = 1$ ;
3.  $\|f(z)\|_X \leq c \frac{1}{(1-|z|^2)^{\alpha-1}} \|f\|_{\mathcal{B}^\alpha(X)}$  for any  $z \in \mathbb{D}$  and  $\alpha > 1$ .

For estimation of  $\|f'(z)\|_X$ ,  $f \in \mathcal{B}^\alpha(X)$ , we have

$$(1 - |z|^2)^\alpha \|f'(z)\|_X \leq \|f(0)\|_X + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \|f'(z)\|_X = \|f\|_{\mathcal{B}^\alpha(X)}.$$

So

$$\|f'(z)\|_X \leq \frac{\|f\|_{\mathcal{B}^\alpha(X)}}{(1 - |z|^2)^\alpha}.$$

In the following theorem we need a differentiation of  $\psi(z)(f(\varphi(z)))$ . We can differentiate  $\psi(z)(f(\varphi(z)))$  similarly as in the scalar-valued case. In fact, by the product rule,

$$(\psi(z)(f(\varphi(z))))' = \psi'(z)(f(\varphi(z))) + \psi(z)(f'(\varphi(z))) \cdot \varphi'(z)$$

which is a  $Y$ -valued analytic function. Note that although for each  $z$ ,  $\psi(z)$  is a linear operator from  $X$  to  $Y$ , the function  $z \mapsto \psi(z)$  is analytic, so it does have a derivative. Indeed, since  $\psi'(z)$  and  $f'(z)$  exist, we have

$$\lim_{w \rightarrow z} \frac{1}{w - z} (\psi(w) - \psi(z))$$

exists in  $L(X, Y)$  and

$$\lim_{w \rightarrow z} \frac{1}{w - z} (f(w) - f(z))$$

exists in  $X$ . Then one must verify that

$$\lim_{w \rightarrow z} \frac{1}{w - z} ((\psi f)(w) - (\psi f)(z))$$

exists in  $X$  with the limit equal to the usual product form.

First, define

$$g_1(\varphi, \psi, z) = \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \|\psi(z)\|_{L(X, Y)} |\varphi'(z)|,$$

$$q_2(\varphi, \psi, z) = (1 - |z|^2) \log \frac{1}{1 - |\varphi(z)|^2} \|\psi(z)\|_{L(X,Y)}.$$

We then set

$$Q_1(\varphi, \psi) = \sup_{z \in \mathbb{D}} q_1(\varphi, \psi, z) \quad \text{and} \quad Q_2(\varphi, \psi) = \sup_{z \in \mathbb{D}} q_2(\varphi, \psi, z).$$

**THEOREM 2.** *Let  $X$  and  $Y$  be complex Banach spaces. Then for  $W_{\psi, \varphi} : \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$ , we have  $W_{\psi, \varphi}$  is bounded if  $Q_1(\varphi, \psi)$  and  $Q_2(\varphi, \psi')$  are finite. Moreover,*

$$\max\{Q_1(\varphi, \psi), Q_2(\varphi, \psi')\} \lesssim \|W_{\psi, \varphi}\| \lesssim \max\{q_2(\varphi, \psi, 0), Q_1(\varphi, \psi), Q_2(\varphi, \psi')\}$$

*Proof.* Suppose  $f \in \mathcal{B}(X)$ . For every  $z \in \mathbb{D}$  we have

$$\begin{aligned} \|(W_{\psi, \varphi}(f)'(z))\|_Y &= \|\psi(z)(f(\varphi(z)))'\|_Y \\ &= \|\psi'(z)(f(\varphi(z))) + \psi(z)(f'(\varphi(z))\varphi'(z))\|_Y \\ &\leq \|\psi'(z)\|_{L(X,Y)} \|f(\varphi(z))\|_X + \|\psi(z)\|_{L(X,Y)} \|f'(\varphi(z))\|_X |\varphi'(z)| \\ &\leq c \log \frac{1}{1 - |\varphi(z)|^2} \|\psi'(z)\|_{L(X,Y)} \|f\|_{\mathcal{B}(X)} + \|\psi(z)\|_{L(X,Y)} \frac{|\varphi'(z)| \|f\|_{\mathcal{B}(X)}}{1 - |\varphi(z)|^2}. \end{aligned}$$

So

$$\begin{aligned} \|W_{\psi, \varphi} f\|_{\mathcal{B}(Y)} &\leq c \log \frac{1}{1 - |\varphi(0)|^2} \|\psi(0)\|_{L(X,Y)} \|f\|_{\mathcal{B}(X)} \\ &\quad + c \sup_{z \in \mathbb{D}} (1 - |z|^2) \log \frac{1}{1 - |\varphi(z)|^2} \|\psi'(z)\|_{L(X,Y)} \|f\|_{\mathcal{B}(X)} \\ &\quad + \sup_{z \in \mathbb{D}} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)| \|\psi(z)\|_{L(X,Y)} \|f\|_{\mathcal{B}(X)}. \end{aligned}$$

This proves the boundedness claim and the upper estimate of the norm. Fix  $x_0 \in X$ . Let  $w \in \mathbb{D}$  and define the functions  $f_w$  as

$$f_w(z) = \frac{1}{\varphi(w)} \left\{ \frac{1 - |\varphi(w)|^2}{1 - \overline{\varphi(w)}z} - 1 \right\} x_0.$$

As in the proof of Theorem 2.1 [16],  $f_w \in \mathcal{B}(X)$  and  $M = \sup\{\|f_w\|_{\mathcal{B}(X)} : w \in \mathbb{D}\} < \infty$ . Also  $f_w(\varphi(w)) = 0$  and  $f'_w(\varphi(w)) = x_0/(1 - |\varphi(w)|^2)$ , which implies  $(W_{\psi, \varphi} f_w)'(w) = \psi(w) f'_w(\varphi(w)) \varphi'(w)$ . So

$$\begin{aligned} M \|W_{\psi, \varphi}\| &\geq \|W_{\psi, \varphi} f_w\|_{\mathcal{B}(Y)} = \sup_{z \in \mathbb{D}} (1 - |z|^2) \|(W_{\psi, \varphi} f_w)'(z)\|_Y \\ &\geq (1 - |w|^2) \|(W_{\psi, \varphi} f_w)'(w)\|_Y \\ &= (1 - |w|^2) \|\psi(w)(f'_w(\varphi(w)))\varphi'(w)\|_Y \\ &= (1 - |w|^2) \|\psi(w)(x_0) \frac{\varphi'(w)}{1 - |\varphi(w)|^2}\|_Y \\ &= \frac{(1 - |w|^2) |\varphi'(w)|}{1 - |\varphi(w)|^2} \|\psi(w)(x_0)\|_Y. \end{aligned}$$

Since  $x_0$  is arbitrary, then

$$\|W_{\psi,\varphi}\| \geq \frac{1}{M} \frac{(1 - |w|^2)|\varphi'(w)|}{1 - |\varphi(w)|^2} \|\psi(w)\|_{L(X,Y)}.$$

Again for  $w \in \mathbb{D}$  being arbitrary, we have

$$\|W_{\psi,\varphi}\| \geq \frac{1}{M} \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} \|\psi(z)\|_{L(X,Y)}. \tag{2.1}$$

Now we define other functions

$$g_w(z) = 2 \log \frac{1}{1 - \overline{\varphi(w)}z} - \left( \log \frac{1}{1 - \overline{\varphi(w)}z} \right)^2 / \log \frac{1}{1 - |\varphi(w)|^2},$$

$$h_w(z) = g_w(z)x_0.$$

By using the same method in [16],  $h_w \in \mathcal{B}(X)$  and  $L = \sup\{\|h_w\|_{\mathcal{B}(X)} : w \in \mathbb{D}\} < \infty$ . Also  $h'_w(\varphi(w)) = 0$  and  $h_w(\varphi(w)) = x_0 \log(1/(1 - |\varphi(w)|^2))$ , which implies  $(W_{\psi,\varphi}f_w)'(w) = \psi'(w)(h_w(\varphi(w)))$ . Then

$$\begin{aligned} L\|W_{\psi,\varphi}\| &\geq \|W_{\psi,\varphi}h_w\|_{\mathcal{B}(Y)} \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \|(W_{\psi,\varphi}h_w)'(z)\|_Y \\ &\geq (1 - |w|^2) \|(W_{\psi,\varphi}h_w)'(w)\|_Y \\ &= (1 - |w|^2) \|\psi'(w)(h_w(\varphi(w)))\|_Y \\ &= (1 - |w|^2) \log \frac{1}{1 - |\varphi(w)|^2} \|\psi'(w)(x_0)\|_Y. \end{aligned}$$

So

$$\|W_{\psi,\varphi}\| \geq \frac{1}{L} \sup_{z \in \mathbb{D}} (1 - |z|^2) \log \frac{1}{1 - |\varphi(z)|^2} \|\psi'(z)\|_{L(X,Y)}. \tag{2.2}$$

The equations (2.1) and (2.2) imply that

$$\begin{aligned} \|W_{\psi,\varphi}\| &\gtrsim \max \left\{ \sup_{z \in \mathbb{D}} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \|\psi(z)\|_{L(X,Y)} |\varphi'(z)| \right. \\ &\quad \left. + \sup_{z \in \mathbb{D}} (1 - |z|^2) \log \frac{1}{1 - |\varphi(z)|^2} \|\psi'(z)\|_{L(X,Y)} \right\}. \quad \square \end{aligned}$$

The above result can be applied to other cases  $0 < \alpha < 1$  and  $\alpha > 1$  between different Bloch spaces  $W_{\psi,\varphi} : \mathcal{B}^\alpha(X) \rightarrow \mathcal{B}^\beta(Y)$ , where  $\beta > 0$ . The proofs are similar.

### 3. (Weak) compactness

A bounded linear operator between two Banach spaces  $X$  and  $Y$  is called compact, weakly compact, if it maps the closed unit ball of  $X$  onto a relatively compact, a relatively weakly compact set in  $Y$ . The class of all (weakly) compact operators between  $X$  and  $Y$  is denoted by  $K(X, Y)$  ( $W(X, Y)$ ). The essential and weak essential norm of an operator  $T : X \rightarrow Y$  are defined by

$$\|T\|_e = \text{dist}(T, K(X, Y)), \quad \|T\|_w = \text{dist}(T, W(X, Y)).$$

The operator  $T$  is compact if and only if  $\|T\|_e = 0$  and is weakly compact if and only if  $\|T\|_w = 0$ .

Here we use the linear operator  $(C_r f)(z) = f(rz)$  for  $f : \mathbb{D} \rightarrow X$  analytic and  $0 < r < 1$ .

LEMMA 3. *The operators  $C_r : \mathcal{B}^\alpha(X) \rightarrow \mathcal{B}^\alpha(X)$  satisfy the following properties.*

1.  $\|C_r\| \leq 1$  for any  $0 < r < 1$ .
2. For every  $R \in (0, 1)$ ,

$$\lim_{r \rightarrow 1} \sup_{\|f\|_{\mathcal{B}^\alpha(X)} \leq 1} \sup_{|z| \leq R} \max\{\|(f - C_r f)'(z)\|_X, \|(f - C_r f)(z)\|_X\} = 0.$$

3. Suppose that  $W_{\psi, \varphi} : \mathcal{B}^\alpha(X) \rightarrow \mathcal{B}^\alpha(Y)$  is bounded. If  $T_\psi : X \rightarrow \mathcal{B}^\alpha(Y)$  is a (weakly) compact operator, then  $W_{\psi, \varphi} \circ C_r : \mathcal{B}^\alpha(X) \rightarrow \mathcal{B}^\alpha(Y)$  is (weakly) compact operator.

*Proof.* For (1), we have

$$\begin{aligned} \|C_r\| &= \sup_{\|f\|_{\mathcal{B}^\alpha(X)} \leq 1} \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \|rf'(rz)\|_X \\ &\leq \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\alpha}{(1 - |rz|^2)^\alpha} \leq \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\alpha}{(1 - |z|^2)^\alpha} = 1. \end{aligned}$$

Let  $0 < r, R < 1$  and  $f : \mathbb{D} \rightarrow X$  be an analytic function and  $z \in \mathbb{D}$ . By setting  $\rho = (|z| + 1)/2$ , we have

$$\begin{aligned} \|f'(z) - rf'(rz)\|_X &= \left\| \int_0^{2\pi} \left( \frac{\rho f'(\rho e^{i\theta})}{\rho - ze^{-i\theta}} - \frac{\rho r f'(\rho e^{i\theta})}{\rho - rze^{-i\theta}} \right) \frac{d\theta}{2\pi} \right\|_X \\ &\leq \sup_{\theta \in [0, 2\pi)} \frac{(1 - r) \|f'(\rho e^{i\theta})\|_X}{|\rho - ze^{-i\theta}| |\rho - rze^{-i\theta}|} \\ &\leq c(1 - r) \frac{\|f\|_{\mathcal{B}^\alpha(X)}}{(1 - |z|)^{2+\alpha}}, \end{aligned}$$

where  $c$  is a positive constant. Moreover, since

$$(f - C_r f)(z) = e^{i\theta} \int_0^{|z|} (f - C_r f)'(te^{i\theta}) dt,$$

where  $z = |z|e^{i\theta}$ , we get

$$\begin{aligned} \|(f - C_r f)(z)\|_X &\leq c(1-r)\|f\|_{\mathcal{B}^\alpha(X)} \int_0^{|z|} \frac{1}{(1-|te^{i\theta}|)^{2+\alpha}} dt \\ &= c(1-r)\|f\|_{\mathcal{B}^\alpha(X)} \int_0^{|z|} \frac{1}{(1-t)^{2+\alpha}} dt \\ &= \frac{c}{2+\alpha-1}(1-r)\|f\|_{\mathcal{B}^\alpha(X)} \left( \frac{1}{(1-|z|)^{2+\alpha-1}} - 1 \right) \end{aligned}$$

Now, the proof of (2) is complete.

Finally, for  $f \in \mathcal{B}^\alpha(X)$  with  $f(z) = \sum_{k=0}^\infty x_k z^k$ , put  $P_n(f) = \sum_{k=0}^n x_k z^k$ ,  $n \geq 0$  and  $q_k(f) = x_k$ ,  $k \in \mathbb{N}$ . Here  $q_k$  are operators  $\mathcal{B}^\alpha(X) \rightarrow X$ . Since  $z^k x_k = \int_0^{2\pi} f(ze^{i\theta}) e^{-ik\theta} \frac{d\theta}{2\pi}$ , then

$$\begin{aligned} \|z^k x_k\|_X &\leq c\|f\|_{\mathcal{B}^\alpha(X)} \quad 0 < \alpha < 1, \\ \|z^k x_k\|_X &\leq c \log \frac{2}{1-|z|^2} \|f\|_{\mathcal{B}^\alpha(X)} \quad \alpha = 1, \\ \|z^k x_k\|_X &\leq c \frac{\|f\|_{\mathcal{B}^\alpha(X)}}{(1-|z|^2)^{\alpha-1}} \quad \alpha > 1, \end{aligned}$$

where  $c > 0$  is a constant. The above relations hold for every  $z \in \mathbb{D}$ . Let  $z \in \mathbb{D}$  and  $|z| = 1/2$ . So

$$\begin{aligned} \|x_k\|_X &\leq c2^k \|f\|_{\mathcal{B}^\alpha(X)} \quad 0 < \alpha < 1, \\ \|x_k\|_X &\leq c2^k \log 8/3 \|f\|_{\mathcal{B}^\alpha(X)} \quad \alpha = 1, \\ \|x_k\|_X &\leq \frac{c2^k 4^{\alpha-1}}{3^{\alpha-1}} \|f\|_{\mathcal{B}^\alpha(X)} \quad \alpha > 1. \end{aligned}$$

Therefore  $q_k$  are bounded on  $\mathcal{B}^\alpha(X)$  in each case. So  $T_\psi \circ q_k : \mathcal{B}^\alpha(X) \rightarrow \mathcal{B}^\alpha(Y)$  are compact operators. Since

$$(W_{\psi,\varphi} P_n f)(z) = \sum_{k=0}^n \varphi(z)^k \cdot \psi(z) x_k = \sum_{k=0}^n \varphi(z)^k \cdot (T_\psi q_k f)(z),$$

it follows that  $W_{\psi,\varphi} P_n$  are compact operators  $\mathcal{B}^\alpha(X) \rightarrow \mathcal{B}^\alpha(Y)$  for all  $n$ . Fix  $0 < r < 1$ , let  $\varepsilon > 0$  and fix  $n_0$  so that  $\sum_{k=n_0+1}^\infty kr^k < \varepsilon$ . Then, for any  $f \in \mathcal{B}^\alpha(X)$  with

$$f(z) = \sum_{k=0}^{\infty} x_k z^k,$$

$$\begin{aligned} \|(C_r - P_{n_0}C_r)f\|_{\mathcal{B}^\alpha(X)} &= \|((C_r - P_{n_0}C_r)f)(0)\|_X \\ &\quad + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \|((C_r - P_{n_0}C_r)f)'(z)\|_X \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \|((C_r - P_{n_0}C_r)f)'(z)\|_X \\ &\leq \|r \sum_{k=n_0+1}^{\infty} k r^k x_k z^{k-1}\|_X < c\mathcal{E}\|f\|_{\mathcal{B}^\alpha(X)} \end{aligned}$$

The above relation holds for every  $n > n_0$ . It follows that  $\|C_r - P_n C_r\| \rightarrow 0$ , as  $n \rightarrow \infty$ . Then  $\|W_{\psi,\varphi}C_r - W_{\psi,\varphi}P_n C_r\| \rightarrow 0$  as  $n \rightarrow \infty$ . Thus  $W_{\psi,\varphi}C_r$  is compact operator. The proof in the case of weakly compact is similar.  $\square$

To characterize compactness, the two conditions we use are

$$\lim_{s \rightarrow 1} \sup_{|\varphi(z)| > s} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \|\psi(z)\|_{L(X,Y)} |\varphi'(z)| = 0 \tag{3.1}$$

$$\lim_{s \rightarrow 1} \sup_{|\varphi(z)| > s} (1 - |z|^2) \log \frac{1}{1 - |\varphi(z)|^2} \|\psi'(z)\|_{L(X,Y)} = 0. \tag{3.2}$$

Weak compactness and compactness of the operator  $W_{\psi,\varphi}$  are related to the operator  $T_\psi$ .

**THEOREM 4.** *Let  $X$  and  $Y$  be complex Banach spaces and  $W_{\psi,\varphi} : \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$  be bounded. Then  $W_{\psi,\varphi} : \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$  is (weakly) compact if and only if*

1.  $T_\psi : X \rightarrow \mathcal{B}(Y)$  is (weakly) compact, and
2. The conditions (3.1) and (3.2) hold.

Since  $T_\psi = W_{\psi,\varphi}A$  where  $A : X \rightarrow \mathcal{B}(X)$  is defined by  $A(x) = f_x$ ,  $f_x(z) = x$ , the (weak compactness) compactness of  $W_{\psi,\varphi}$  implies  $T_\psi$  is (weakly) compact.

*Proof of sufficiency in Theorem 4.* Suppose that  $T_\psi : X \rightarrow \mathcal{B}(Y)$  is weakly compact and (3.1) and (3.2) hold. It will be enough to prove that

$$\begin{aligned} \|W_{\psi,\varphi}\|_w &\leq 2 \lim_{s \rightarrow 1} \sup_{|\varphi(z)| > s} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \|\psi(z)\|_{L(X,Y)} |\varphi'(z)| \\ &\quad + 2 \lim_{s \rightarrow 1} \sup_{|\varphi(z)| > s} (1 - |z|^2) \log \frac{1}{1 - |\varphi(z)|^2} \|\psi'(z)\|_{L(X,Y)}. \end{aligned}$$

By Lemma 3(3), the operators  $W_{\psi,\varphi}C_r$  are weakly compact too. So

$$\|W_{\psi,\varphi}\|_w \leq \|W_{\psi,\varphi} - W_{\psi,\varphi}C_r\|.$$



For every  $f \in \mathcal{B}(X)$  and  $z \in \mathbb{D}$  we have

$$\begin{aligned} & (1 - |z|^2) \|(W_{\psi, \varphi} f - W_{\psi, \varphi} C_r f)'(z)\|_Y \\ & \leq \max\left\{ \sup_{|\varphi(z)| > s} (1 - |z|^2) \|(W_{\psi, \varphi} f - W_{\psi, \varphi} C_r f)'(z)\|_Y, \right. \\ & \quad \left. \sup_{|\varphi(z)| \leq s} (1 - |z|^2) \|(W_{\psi, \varphi} f - W_{\psi, \varphi} C_r f)'(z)\|_Y \right\} \\ & = \max\{I, J\}. \end{aligned}$$

For  $I$ , using Lemma 1 we have

$$\begin{aligned} I & \leq \sup_{|\varphi(z)| > s} (1 - |z|^2) \|\psi'(z)(f(\varphi(z))) - \psi'(z)(C_r(f(\varphi(z))))\|_Y \\ & \quad + \sup_{|\varphi(z)| > s} \|\psi(z)(f'(\varphi(z)))\varphi'(z) - \psi(z)((C_r f)'(\varphi(z)))\varphi'(z)\|_Y \\ & \leq \sup_{|\varphi(z)| > s} (1 - |z|^2) \|\psi'(z)\| \|f(\varphi(z)) - C_r(f(\varphi(z)))\|_X \\ & \quad + \sup_{|\varphi(z)| > s} (1 - |z|^2) \|\psi(z)\| \|f'(\varphi(z))\varphi'(z) - (C_r f)'(\varphi(z))\varphi'(z)\|_X \\ & \leq 2 \sup_{|\varphi(z)| > s} (1 - |z|^2) \log \frac{1}{1 - |\varphi(z)|^2} \|\psi'(z)\|_{L(X, Y)} \|f\|_{\mathcal{B}(X)} \\ & \quad + 2 \sup_{|\varphi(z)| > s} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \|\psi(z)\|_{L(X, Y)} |\varphi'(z)| \|f\|_{\mathcal{B}(X)}. \end{aligned}$$

Put

$$E = \sup_{z \in \mathbb{D}} (1 - |z|^2) \|\psi'(z)\|_{L(X, Y)} \quad \text{and} \quad F = \sup_{z \in \mathbb{D}} (1 - |z|^2) \|\psi(z)\|_{L(X, Y)} |\varphi'(z)|.$$

To see that  $E$  and  $F$  are finite, take  $f_1(z) = x_0$  and  $f_2(z) = x_0 z$ , where  $x_0 \in X$  is arbitrary with  $\|x_0\|_X = 1$ . Then using the boundedness of  $W_{\psi, \varphi}$  implies the results. For  $J$ ,

$$\begin{aligned} J & \leq \sup_{\|f\|_{\mathcal{B}(X)} \leq 1} \sup_{|\varphi(z)| \leq s} (1 - |z|^2) \|\psi'(z)\|_{L(X, Y)} \|f(\varphi(z)) - C_r(f(\varphi(z)))\|_X \\ & \quad + \sup_{\|f\|_{\mathcal{B}(X)} \leq 1} \sup_{|\varphi(z)| \leq s} (1 - |z|^2) \|\psi(z)\|_{L(X, Y)} \|f'(\varphi(z))\varphi'(z) - (C_r f)'(\varphi(z))\varphi'(z)\|_X \\ & \leq E \sup_{\|f\|_{\mathcal{B}(X)} \leq 1} \sup_{|\varphi(z)| \leq s} \|f(\varphi(z)) - C_r(f(\varphi(z)))\|_X \\ & \quad + F \sup_{\|f\|_{\mathcal{B}(X)} \leq 1} \sup_{|\varphi(z)| \leq s} \|f'(\varphi(z)) - (C_r f)'(\varphi(z))\|_X. \end{aligned}$$

Letting  $r \rightarrow 1$ , according to Lemma 3(2), we have

$$\lim_{r \rightarrow 1} J = 0. \quad \square$$

The idea for the proof of the necessity part is to use a Leibov-type argument similar to the one in [10].

LEMMA 5. *Let  $\{f_n\}$  be a sequence in  $\mathcal{B}_0(X)$ ,  $\|f_n\|_{\mathcal{B}(X)} = 1$  and  $\|f_n\|_{B_2(X)} \rightarrow 0$  as  $n \rightarrow \infty$ . Then there exists a subsequence  $\{f_{n_k}\}$  such that it is equivalent to the natural basis of  $c_0$ , that is, the map  $(\lambda_k) \mapsto \sum_k \lambda_k f_{n_k}$  is an isomorphism from  $c_0$  into  $\mathcal{B}_0(X)$ .*

*Proof.* It can be proved that  $\|f_n\|_{\mathcal{B}(X)} \leq \sup_{|x^*| \leq 1} \|x^* \circ f_n\|_{\mathcal{B}}$ . So there exists an  $x^* \in X^*$  such that  $\|f_n\|_{\mathcal{B}(X)} \leq \|x^* \circ f_n\|_{\mathcal{B}}$ . On the other hand  $\|x^* \circ f_n\|_{\mathcal{B}} \leq \|f_n\|_{\mathcal{B}(X)}$ . Now we define  $g_n = x^* \circ f_n$ . Then  $g_n \in \mathcal{B}_0$ ,  $\|g_n\|_{\mathcal{B}} = 1$  and  $\|g_n\|_{B_2} \rightarrow 0$  as  $n \rightarrow \infty$ . Set

$$\gamma(g_n, a) = \|g_n \circ \varphi_a - g_n(a)\|_{B_2}.$$

The change of variable implies that  $\gamma(g_n, a) \leq \|g_n \circ \varphi_a\|_{B_2} \leq c_a \|g_n\|_{B_2}$  where  $c_a$  is an increasing function of  $|a|$ . Fix any  $0 < r < 1$  and  $|a| \leq r$ ,  $c_a \leq c_r$ . It follows from  $\|g_n\|_{B_2} \rightarrow 0$  that

$$\sup_{|a| \leq r} \gamma(g_n, a) \rightarrow 0$$

as  $n \rightarrow \infty$ . Since  $\{g_n\} \in \mathcal{B}_0$ ,  $\gamma(g_n, a) \rightarrow 0$  as  $|a| \rightarrow 1$  for each  $n$ . These properties imply that there exist increasing sequences of positive integers  $\{n_k\}$  and numbers  $0 < r_k < 1$  such that for each  $k$ ,  $\|g_{n_k}\|_{B_2} < 2^{-k-1}$  and

$$\sup_{|a| \leq r_k} \gamma(g_{n_k}, a) < 2^{-k-1}, \quad \sup_{|a| \geq r_{k+1}} \gamma(g_{n_k}, a) < 2^{-k-1}.$$

For every  $a \in \mathbb{D}$  we then have  $\gamma(g_{n_k}, a) < 2^{-k-1}$  for all but possibly one index  $k$ , for which  $\gamma(g_{n_k}, a) \leq 1$ . Hence  $\sum_k \gamma(g_{n_k}, a) < 1 + \frac{1}{2} = \frac{3}{2}$ . Now we define the map  $S : c_0 \rightarrow \mathcal{B}_0$  by

$$S\lambda = \sum_{k=1}^{\infty} \lambda_k g_{n_k}$$

for  $\lambda = (\lambda_k) \in c_0$ . Since  $\|\sum_{k=1}^{\infty} \lambda_k g_{n_k}\|_{B_2} \leq \sum_{k=1}^{\infty} |\lambda_k| \|g_{n_k}\|_{B_2}$ , the series converges in  $B_p$ . It can be seen from the inequality

$$\gamma(S\lambda, a) \leq \sum_{k=1}^{\infty} |\lambda_k| \gamma(g_{n_k}, a) \leq \frac{3}{2} \|\lambda\|_{\infty}$$

that  $\|S\lambda\|_{\mathcal{B}} \leq \frac{3}{2} c \|\lambda\|_{\infty}$ . Since  $\lambda \in c_0$ , we have  $\lambda_k \rightarrow 0$ . So there exists an integer  $K$  such that  $|\lambda_k| < \varepsilon$  for  $k > K$ . Then

$$\begin{aligned} \gamma(S\lambda, a) &\leq \sum_{k=1}^{\infty} |\lambda_k| \gamma(g_{n_k}, a) \\ &= \sum_{k=1}^K |\lambda_k| \gamma(g_{n_k}, a) + \sum_{k=K+1}^{\infty} |\lambda_k| \gamma(g_{n_k}, a) \\ &\leq \|\lambda\|_{\infty} \sum_{k=1}^K |\lambda_k| \gamma(g_{n_k}, a) + \frac{3}{2} \varepsilon. \end{aligned}$$

Since  $\gamma(g_{n_k}, a) \rightarrow 0$  as  $|a| \rightarrow 1$  for each  $k$ , and  $\varepsilon > 0$  was arbitrary, this implies that  $S\lambda \in \mathcal{B}_0$ . Therefore  $S$  is a bounded linear operator from  $c_0$  into  $\mathcal{B}_0$ .

To check that  $S$  is one to one, we prove that  $S$  is bounded below. For  $\lambda = (\lambda_k) \in c_0$ , there exists an index  $K$  for which  $|\lambda_K| = \|\lambda\|_\infty$ . We know that  $|g_{n_K}(0)| \leq \|g_{n_K}\|_{B_2} < \frac{1}{4}$  and  $\|g_{n_K}\|_{\mathcal{B}} = 1$ . So there exists a positive constant  $c$  such that  $\sup_{a \in \mathbb{D}} \gamma(g_{n_K}, a) \approx 1 - |g_{n_K}(0)| > 1 - \frac{1}{4}$ . Hence there is a point  $a \in \mathbb{D}$  such that  $\gamma(g_{n_K}, a) > 1 - \frac{1}{4}$ . Note that for  $k \neq K$ , we have  $\gamma(g_{n_k}, a) < 2^{-k-1}$ . Therefore

$$\begin{aligned} \|S\lambda\|_{\mathcal{B}} &\geq c\gamma(S\lambda, a) \gtrsim |\lambda_K|\gamma(g_{n_K}, a) - \sum_{k \neq K} |\lambda_k|\gamma(g_{n_k}, a) \\ &\geq \left(1 - \frac{1}{4}\right) \|\lambda\|_\infty - \frac{1}{2}\|\lambda\|_\infty = \frac{1}{4}\|\lambda\|_\infty. \end{aligned}$$

We have proved that  $S$  is an isomorphism from  $c_0$  into  $\mathcal{B}_0$ . An easy calculation shows that  $S\lambda = x^*(T\lambda)$  where  $T\lambda = \sum_{k=1}^\infty \lambda_k f_{n_k}$ . Then  $T$  is an isomorphism from  $c_0$  into  $\mathcal{B}_0(X)$  and we are done.  $\square$

*Proof of necessity in Theorem 4.* Suppose that the conditions (3.1) and (3.2) fail. We will complete the proof by proving that  $W_{\psi, \varphi} : \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$  fixes a copy of  $c_0$  and therefore it is not weakly compact.

If the condition (3.1) fails, then there exist  $c > 0$  and a sequence  $\{a_n\} \in \mathbb{D}$  such that  $|\varphi(a_n)| \rightarrow 1$  as  $n \rightarrow \infty$  and

$$\frac{1 - |a_n|^2}{1 - |\varphi(a_n)|^2} \|\psi(a_n)\|_{L(X,Y)} |\varphi'(a_n)| > c.$$

Let  $x \in X$  with  $\|x\|_X = 1$ . Define the functions  $f_n$  by  $f_n(z) = g_n(z)x$  where

$$g_n(z) = \frac{(1 - |\varphi(a_n)|^2)^2}{(1 - \overline{\varphi(a_n)}z)^2} - \frac{1 - |\varphi(a_n)|^2}{1 - \overline{\varphi(a_n)}z}.$$

Then  $M = \|f_n\|_{\mathcal{B}(X)} < \infty$ ,  $f_n(\varphi(a_n)) = 0$  and  $f'_n(\varphi(a_n)) = \overline{\varphi(a_n)}/(1 - |\varphi(a_n)|^2)x$ . Furthermore  $f_n \in \mathcal{B}_0(X)$  and

$$\begin{aligned} \|f_n\|_{B_2(X)}^2 &= \int_{\mathbb{D}} \left\| \frac{(1 - |\varphi(a_n)|^2)^2}{(1 - \overline{\varphi(a_n)}z)^2} - \frac{1 - |\varphi(a_n)|^2}{1 - \overline{\varphi(a_n)}z} x \right\|_X^2 dA(z) \\ &\leq 4 \int_{\mathbb{D}} \frac{(1 - |\varphi(a_n)|^2)^4}{|1 - \overline{\varphi(a_n)}z|^4} + \frac{(1 - |\varphi(a_n)|^2)^2}{|1 - \overline{\varphi(a_n)}z|^2} dA(z) \\ &= 4(1 - |\varphi(a_n)|^2)^4 \int_{\mathbb{D}} \frac{1}{|1 - \overline{\varphi(a_n)}z|^4} dA(z) \\ &\quad + 4(1 - |\varphi(a_n)|^2)^2 \int_{\mathbb{D}} \frac{1}{|1 - \overline{\varphi(a_n)}z|^2} dA(z) \\ &\leq 4c(1 - |\varphi(a_n)|^2)^2 + 4c(1 - |\varphi(a_n)|^2)^2 \log \frac{1}{1 - |\varphi(a_n)|^2}. \end{aligned}$$

The last line of the above relation is due to Theorem 1.12 of [20]. So  $\|f_n\|_{B_2(X)} \rightarrow 0$  as  $n \rightarrow \infty$ . Also

$$\begin{aligned} \|W_{\psi,\varphi}f_n\|_{\mathcal{B}(X)} &\geq (1 - |a_n|^2)\|W_{\psi,\varphi}(f_n)'(a_n)\|_X \\ &= (1 - |a_n|^2)\|\psi(a_n)(f_n'(\varphi(a_n)))\varphi'(a_n)\|_X \\ &= \frac{1 - |a_n|^2}{1 - |\varphi(a_n)|^2}\|\psi(a_n)(x)\|_X|\varphi'(a_n)||\varphi(a_n)|, \end{aligned}$$

for every  $x \in X$ . So

$$\|W_{\psi,\varphi}f_n\|_{\mathcal{B}(X)} \geq \frac{1 - |a_n|^2}{1 - |\varphi(a_n)|^2}\|\psi(a_n)\|_{L(X,Y)}|\varphi'(a_n)||\varphi(a_n)| \geq \frac{c}{2}.$$

For using Lemma 5, we should have  $\|f_n\|_{\mathcal{B}(X)} = 1$ . This can be done by normalizing the norm of the sequence  $\{f_n\}$ . Without loss of generality we use  $\{f_n\}$  again. So we can find a subsequence  $\{f_{n_k}\}$  which is equivalent to the natural basis of  $c_0$  which implies that  $\{W_{\psi,\varphi}f_{n_k}\}$  is a weak-null sequence in  $\mathcal{B}(X)$ . Using the Bessaga-Polczynski selection principle (see [1, 1.3.10]) to  $\{W_{\psi,\varphi}f_{n_k}\}$ , there exists a subsequence, say  $\{f_{n_k}\}$  again, such that  $\{W_{\psi,\varphi}f_{n_k}\}$  is a semi-normalized basic sequence in  $\mathcal{B}(X)$ . Hence there are constants  $A, B > 0$  such that

$$\begin{aligned} A \cdot \|\lambda\|_\infty &\leq \left\| \sum_{k=1}^\infty \lambda_k W_{\psi,\varphi}f_{n_k} \right\|_{\mathcal{B}(X)} \leq \|W_{\psi,\varphi}\| \cdot \left\| \sum_{k=1}^\infty \lambda_k f_{n_k} \right\|_{\mathcal{B}(X)} \\ &\leq B \cdot \|W_{\psi,\varphi}\| \|\lambda\|_\infty, \end{aligned}$$

for every  $\lambda = (\lambda_k) \in c_0$ . These estimates state that the restriction of  $W_{\psi,\varphi}$  to the closed subspace of  $\mathcal{B}(X)$  spanned by the sequence  $\{f_{n_k}\}$  is an isomorphism onto a linearly isomorphic copy of  $c_0$ , and we are done.

If condition (3.2) fails, then we have the same result by using the functions

$$g_n(z) = \frac{-1}{\log(1 - |\varphi(z_n)|^2)} \left( 3 \left( \log \frac{1}{1 - \varphi(z_n)z} \right)^2 - 2 \left( \log \frac{1}{1 - \varphi(z_n)z} \right)^3 \right). \quad \square$$

#### 4. Compactness properties of $T_\psi$

EXAMPLE 6. There is an analytic operator-valued map  $\psi \in H^\infty(L(\ell^1))$ ,  $\psi(z) \in K(\ell^1)$ , but  $T_\psi : \ell^1 \rightarrow \mathcal{B}(\ell^1)$  is not even weakly conditionally compact.

*Proof.* Define the bounded operator-valued analytic map  $\psi : \mathbb{D} \rightarrow L(\ell^1)$  by  $\psi(z) = \sum_{k=1}^\infty z^k e_k^* \otimes e_k$ , where  $(e_k)$  denotes the standard unit vector basis of  $\ell^1$  and  $(e_k^*) \subset c_0$  its biorthogonal sequence. In other words,

$$\psi(z)x = \sum_{k=1}^\infty z^k x_k e_k, \quad x = (x_k) \in \ell^1, z \in \mathbb{D}.$$

The claim is that  $T_\Psi$  is not weakly conditionally compact as an operator  $\ell^1 \rightarrow \mathcal{B}(\ell^1)$ . Suppose to the contrary that  $T_\Psi$  is weakly conditionally compact. So there exists a weakly Cauchy subsequence  $(T_\Psi(e_{n_j}))$  such that the difference sequence  $(T_\Psi(e_{n_{2j+1}} - e_{n_{2j}}))$  is weak-null in  $\mathcal{B}(\ell^1)$ . By Mazur's theorem,

$$\left\| \sum_{j=1}^s c_j T_\Psi(e_{n_{2j+1}} - e_{n_{2j}}) \right\|_{\mathcal{B}(\ell^1)} < \frac{1}{2}$$

for a suitable convex combination, where  $\sum_{j=1}^s c_j = 1$  and  $c_j \geq 0$  for  $j = 1, \dots, s$ . On the other hand, we have

$$\begin{aligned} & \left\| \sum_{j=1}^s c_j T_\Psi(e_{n_{2j+1}} - e_{n_{2j}}) \right\|_{\mathcal{B}(\ell^1)} \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \left\| \sum_{j=1}^s c_j (n_{2j+1} z^{n_{2j+1}-1} e_{n_{2j+1}} - n_{2j} z^{n_{2j}-1} e_{n_{2j}}) \right\|_{\ell^1} \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \sum_{j=1}^s c_j (n_{2j+1} |z|^{n_{2j+1}-1} + n_{2j} |z|^{n_{2j}-1}) \\ &\geq \sum_{j=1}^s c_j (n_{2j+1} + n_{2j}) \geq 1, \end{aligned}$$

which is a contradiction.  $\square$

**LEMMA 7.** *Let  $X$  be a complex Banach spaces,  $X_0 \subset X$  be a closed subspace and  $f \in \mathcal{B}(X)$ . Then  $f \in \mathcal{B}_0(X_0)$  if and only if  $f \in \mathcal{B}_0(X)$  and  $f(\mathbb{D}) \subset X_0$ .*

*Proof.* It is obvious that if  $f \in \mathcal{B}_0(X_0)$  then  $f \in \mathcal{B}_0(X)$  and  $f(\mathbb{D}) \subset X_0$ . Suppose that  $f \in \mathcal{B}_0(X)$  and  $f(\mathbb{D}) \subset X_0$ . Define  $f_r(z) = f(rz)$ ,  $0 < r < 1$ . So

$$\begin{aligned} \|f_r\|_{\mathcal{B}(X_0)} &= \|f(0)\|_{X_0} + \sup_{z \in \mathbb{D}} (1 - |z|^2) \|rf'(rz)\|_{X_0} \\ &< \|f(0)\|_{X_0} + \sup_{z \in \mathbb{D}} (1 - |z|^2) \|f'(rz)\|_{X_0} \\ &< \|f(0)\|_{X_0} + \sup_{z \in \mathbb{D}} (1 - |rz|^2) \|f'(rz)\|_X \leq \|f\|_{\mathcal{B}(X)}. \end{aligned}$$

It means that  $f_r \in \mathcal{B}(X_0)$ . Then

$$\begin{aligned} \lim_{|z| \rightarrow 1} (1 - |z|^2) \|f'_r(z)\|_{X_0} &= \lim_{|z| \rightarrow 1} (1 - |z|^2) \|rf'(rz)\|_{X_0} \\ &< \lim_{|z| \rightarrow 1} (1 - |z|^2) \|f'(rz)\|_{X_0} \\ &< \lim_{|z| \rightarrow 1} (1 - |rz|^2) \|f'(rz)\|_X = 0. \end{aligned}$$

Hence  $f_r \in \mathcal{B}_0(X_0)$ . There are polynomials  $p_n(z) = \sum_{j=0}^{N_n} z^j x_j^{(n)}$  such that  $p_n \rightarrow f$  in  $\mathcal{B}(X)$  as  $n \rightarrow \infty$ . By using the same way we have  $(p_n)_r \rightarrow f_r$  as  $n \rightarrow \infty$ . Also

$$\begin{aligned} \|p_n - (p_n)_r\|_{\mathcal{B}(X)} &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \left\| \sum_{j=1}^{N_n} j z^{j-1} x_j^{(n)} (1 - r^j) \right\|_X \\ &< \sup_{z \in \mathbb{D}} (1 - |z|^2) \left( \sup_{1 \leq j \leq N_n} \|x_j^{(n)}\|_X \right) N_n \sum_{j=1}^{N_n} (1 - r^j) \rightarrow 0, \end{aligned}$$

as  $r \rightarrow 1$ . Since

$$\begin{aligned} \|f - f_r\|_{\mathcal{B}(X_0)} &= \|f - f_r\|_{\mathcal{B}(X)} \\ &\leq \|f - p_n\|_{\mathcal{B}(X)} + \|p_n - (p_n)_r\|_{\mathcal{B}(X)} + \|(p_n)_r - f_r\|_{\mathcal{B}(X)}, \end{aligned}$$

we deduce that  $f_r \rightarrow f$  in  $\mathcal{B}_0(X_0)$  as  $r \rightarrow 1$ .  $\square$

**THEOREM 8.** *Let  $X$  and  $Y$  be complex Banach spaces and  $\psi \in \mathcal{B}_0(L(X, Y))$ . Then  $T_\psi : X \rightarrow \mathcal{B}(Y)$  is compact (weakly compact) if and only if  $\psi(\mathbb{D}) \subset K(X, Y)$  ( $\psi(\mathbb{D}) \subset W(X, Y)$ ).*

*Proof.* Suppose that  $\psi(\mathbb{D}) \subset K(X, Y)$ . The previous lemma implies that  $\psi \in \mathcal{B}_0(K(X, Y))$ . Find  $K(X, Y)$ -valued polynomials  $\psi_n(z) = \sum_{k=0}^n z^k U_k^{(n)}$  such that  $\psi_n \rightarrow \psi$  in  $\mathcal{B}(K(X, Y))$ . Since  $\|T_\psi\| = \|\psi\|_{\mathcal{B}(L(X, Y))}$ , we have  $\|T_\psi - T_{\psi_n}\| \rightarrow 0$  as  $n \rightarrow \infty$ . So it will be sufficient to prove that  $T_{\psi_n} : X \rightarrow \mathcal{B}(Y)$  is compact. Define the maps  $\theta_k : Y \rightarrow \mathcal{B}(Y)$  by  $(\theta_k y)(z) = z^k y$ . Each  $\theta_k$  is bounded, then  $\theta_k \circ U_k^{(n)}$  is compact and so  $T_{\psi_n} = \sum_{k=0}^n \theta_k \circ U_k^{(n)}$ .

Now, suppose that  $T_\psi : X \rightarrow \mathcal{B}(Y)$  is compact. Fix  $z \in \mathbb{D}$  and define  $\gamma : \mathcal{B}(Y) \rightarrow Y$  by  $\gamma(f) = f(z)$ . Then  $\gamma$  is a bounded linear operator and  $\gamma \circ T_\psi = \psi(z)$ . So  $\psi(z) : X \rightarrow Y$  is a compact operator.  $\square$

**EXAMPLE 9.** (1) Let  $X$  be any Banach space,  $\psi(z) \equiv U$  and  $\varphi(z) = \frac{z+1}{2}$  for  $z \in \mathbb{D}$ , where  $U \in K(X)$  is a fixed operator. Then  $W_{\psi, \varphi}$  is compact  $\mathcal{B}(X) \rightarrow \mathcal{B}(X)$ . Indeed

$$\begin{aligned} \lim_{s \rightarrow 1} \sup_{|\varphi(z)| > s} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \|\psi(z)\|_{L(X, Y)} |\varphi'(z)| &\leq \frac{1}{2} \|U\|_{L(X, Y)} \lim_{s \rightarrow 1} \sup_{|\varphi(z)| > s} \frac{1 - |z|^2}{-|z|^2} = 0, \\ \lim_{s \rightarrow 1} \sup_{|\varphi(z)| > s} (1 - |z|^2) \log \frac{1}{1 - |\varphi(z)|^2} \|\psi'(z)\|_{L(X, Y)} &= 0. \end{aligned}$$

Also  $T_\psi$  is compact by Theorem 8 since  $\psi(z) \equiv U \in K(X)$ . So Theorem 4 implies that  $W_{\psi, \varphi}$  is compact.

(2) Let  $X$  be any reflexive Banach space,  $\psi(z) \equiv V$  and  $\varphi(z) = \frac{z+1}{2}$  for  $z \in \mathbb{D}$ , where  $V \notin K(X)$  is a fixed operator. Then  $W_{\psi, \varphi}$  is weakly compact, but not compact. Non-compactness of  $W_\psi$  is because of non-compactness of  $T_\psi$  (Theorem 8,  $\psi(z) \equiv V \notin K(X)$ ). Since  $X$  is reflexive,  $V$  is weakly compact. So  $T_\psi$  is weakly compact by Theorem 8. Also the conditions (3.1) and (3.2) hold. Now  $W_{\psi, \varphi}$  is weakly compact.

*Acknowledgement.* The author would like to thank professor Jussi Laitila for giving the idea of this paper and for valuable comments. Also I wish to thank the referee for the time spent studying the manuscript and for his/her suggestions which improved the manuscript.

## REFERENCES

- [1] F. ALBIAC AND N. J. KALTON, *Topics in Banach Space Theory*, Springer, 2006.
- [2] J. BONET, P. DOMANSKI AND M. LINDSTROM, *Weakly compact composition operators on analytic vector-valued function spaces*, Ann. Acad. Sci. Fenn. Math. **26** (2001), 233–248.
- [3] J. BONET, M. C. GÓMEZ-COLLADO, D. JORNET AND E. WOLF, *Operator-weighted composition operators between weighted spaces of vector-valued analytic functions*, Ann. Acad. Sci. Fenn. Math. **37** (2012), 319–338.
- [4] C. C. COWEN AND B. D. MACCLUER, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.
- [5] J. DIESTEL, H. JARCHOW AND A. TONGE, *Absolutely Summing Operators*, Cambridge Univ. Press, Cambridge 1995.
- [6] M. HASSANLOU, J. LAITILA AND H. VAEZI, *Weighted composition operators between weak spaces of vector-valued analytic functions*, Analysis, **37** (1) (2017), 39–45.
- [7] E. HILLE AND R. S. PHILLIPS, *Functional Analysis and Semigroups*, Amer. Math. Soc. Providence, R. I. 1957.
- [8] J. LAITILA, *Weakly compact composition operators on vector-valued BMOA*, J. Math. Anal. Appl. **308** (2005), 730–745.
- [9] J. LAITILA, *Composition operators and vector-valued BMOA*, Integr. Equ. Oper. Theory, **58** (2007), 487–502.
- [10] J. LAITILA, P. J. NIEMINEN, E. SAKSMAN AND H.-O. TYLLI, *Compact and weakly compact composition operators on BMOA*, Complex Analysis and Operator Theory **7** (2013), 163–181.
- [11] J. LAITILA AND H.-O. TYLLI, *Operator-weighted composition operators on vector-valued analytic function spaces*, Illinois J. Math. **53** (2009), 1019–1032.
- [12] J. LAITILA, H.-O. TYLLI, AND M. WANG, *Composition operators from weak to strong spaces of vector-valued analytic functions*, J. Operator Theory **62** (2009), 281–295.
- [13] P. LIU, E. SAKSMAN AND H.-O. TYLLI, *Small composition operators on analytic vector-valued function spaces*, Pacific J. Math. **184** (1998), 295–309.
- [14] J. S. MANHAS, *Weighted composition operators on weighted spaces of vector-valued analytic functions*, J. Korean Math. Soc. **45** (2008), 1203–1220.
- [15] M. NOWAK, *On Bloch space in the unit ball of  $\mathbb{C}^n$* , Ann. Acad. Sci. Fenn. **23** (1998), 461–473.
- [16] S. OHNO, K. STROETHOFF AND R. ZHAO, *Weighted composition operators between Bloch-type spaces*, Rocky Mountain J. Math. **33** (1) (2003), 191–215.
- [17] J.H. SHAPIRO, *Composition operator and classical function theory*, Springer-Verlag, New York, 1993.
- [18] M. WANG, *Weighted composition operators between Dirichlet spaces*, Acta Math. Sci. 2011, 31B (2): 641–651.
- [19] M. WANG, *Riemann-Stieltjes operators between vector-valued weighted Bloch spaces*, J. Ineq. Appl., (2008), doi:10.1155/2008/348208.
- [20] K. ZHU, *Spaces of holomorphic functions in the unit ball*, Springer, New York, 2005.

(Received February 21, 2018)

Mostafa Hassanlou  
Engineering Faculty of Koy  
Urmia University  
Urmia, Iran  
e-mail: m.hassanlou@urmia.ac.ir