

COMPLEMENTARITY OF SUBSPACES OF ℓ_∞ REVISITED

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Abstract. We present a simple criterion for complementarity of subspaces of ℓ_∞ induced by certain bounded linear operators. As applications, it is shown that some typical and well-known subspaces such as mean or almost convergent sequence spaces are uncomplemented in ℓ_∞ . We also note that there exists a weak* closed uncomplemented subspace of ℓ_∞ .

1. Introduction

Let ℓ_∞ , c and c_0 denote the Banach spaces of bounded, convergent or null sequences, respectively. The first example of an uncomplemented subspace of ℓ_∞ is c (and c_0). This folklore result was given by Phillips in his 1940 paper [14]. The original proof is based on a detailed study of representation of linear operators. Nearly a quarter century later, Whitley [16] drastically simplified the proof of Phillips' result by using an idea due to Nakamura and Kakutani [13]. Precisely, he showed that $(\ell_\infty/c_0)^*$ has no countable total subset; and it suffices to conclude that c_0 is not complemented in ℓ_∞ since the property that the dual space has a countable total subset is preserved under taking subspaces or by linear isomorphisms.

Complementarity of subspaces of ℓ_∞ had been deeply studied as a part of the main stream of the isomorphic theory. In 1967, Lindenstrauss [10] gave an important characterization of complemented subspaces of ℓ_∞ by showing that ℓ_∞ is a prime Banach space, that is, an infinite dimensional complemented subspace of ℓ_∞ must be isomorphic to ℓ_∞ . (The converse implication follows from the fact that ℓ_∞ is an injective Banach space; see, for example, [1, Proposition 2.5.2].) It follows that, at least, there is no separable infinite dimensional complemented subspace of ℓ_∞ . At this point, Phillips' result was significantly improved by Lindenstrauss.

The theoretical development for the study of Banach space structure of complemented subspaces of ℓ_∞ has been mostly reached the stage of satisfaction (since such spaces are isomorphically "the same" as ℓ_∞). However, this well-known characterization is not always effective in determining the complementarity of concrete non-separable subspaces of ℓ_∞ . To do this, we still have to investigate for case by case; because we do not know whether checking an infinite dimensional subspace of ℓ_∞ is

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(not) isomorphic to ℓ_∞ is easier than examining the complementarity of the subspace directly.

In this paper, we present a simple criterion for complementarity of subspaces of ℓ_∞ induced by bounded linear operators admitting matrix representations. The proof employs the above mentioned argument of Whitley, that is, we check whether the dual spaces of the quotient of ℓ_∞ by such subspaces have countable total subsets. As an application, among other examples, we show that closed subspaces between c_0 and the mean convergent sequence space are all uncomplemented in ℓ_∞ . From this and Lorentz’s theorem [11], in particular, we conclude that the space of almost convergent sequences is also uncomplemented in ℓ_∞ . On the other hand, we provide an example of a weak* closed uncomplemented subspace of ℓ_∞ . It is known that if M is a weak* closed subspace of ℓ_∞ then $(\ell_\infty/M)^*$ always has a countable total subset. Consequently, we see that there is a limit to determining the complementarity of subspaces of ℓ_∞ by using Whitley’s method.

2. Subspaces of ℓ_∞ induced by matrices

Let $B(\ell_\infty)$ denote the Banach space of bounded linear operators on ℓ_∞ . Suppose that $T \in B(\ell_\infty)$. We consider the closed subspaces $c(T) := T^{-1}(c)$ and $c_0(T) := T^{-1}(c_0)$ of ℓ_∞ , respectively. We note that $c(I) = c$ and $c_0(I) = c_0$ while $c(0) = c_0(0) = \ell_\infty$.

A linear operator T on ℓ_∞ is said to *admits a matrix representation* if there exists an infinite matrix (t_{ij}) of complex numbers such that $(Ta)_n = \sum_{j=1}^\infty t_{nj}a_j$ for each $a = (a_n) \in \ell_\infty$. If $T \in B(\ell_\infty)$ admits a matrix representation, the spaces $c(T)$ and $c_0(T)$ are closely related to objects studied in the monograph [4]. In particular, $c(T)$ is called the *bounded summability field* of T ; see also [5, 7]. For further information of the “summability domains” of matrices in normed spaces and the matrix transformations, the readers are referred to [2].

Let X be a Banach space. A subset F of X^* is said to be *total* if $f(x) = 0$ for each $f \in F$ implies that $x = 0$. Now suppose that M is a subspace of X , and that Y is a Banach space isomorphic to X . If X^* has a countable total subset then M^* and Y^* also have countable total subsets. Since ℓ_∞^* has a countable total subset consisting of coordinate functionals, it follows that each complemented subspace of ℓ_∞ must have such a set.

Now we present the main theorem. The proof is based on a combination of a *gliding hump argument* and Whitley’s method [16].

THEOREM 2.1. *Let $T \in B(\ell_\infty)$ with a matrix representation (t_{ij}) . Suppose that $c_0 \subset c_0(T) \subsetneq \ell_\infty$. If M is a closed subspace with $c_0 \subset M \subset c(T)$, then $(\ell_\infty/M)^*$ has no countable total subsets. Consequently, M is not complemented in ℓ_∞ .*

Proof. Let $e_n = (0, \dots, 0, 1, 0, \dots)$ for each $n \in \mathbb{N}$, where 1 is in the n -th position; and let $e_n^*a = a_n$ for each $n \in \mathbb{N}$ and each $a = (a_n) \in \ell_\infty$. We note that $t_{ij} = e_i^*Te_j$ for each $i, j \in \mathbb{N}$. Let γ_{ij} be a complex number such that $|\gamma_{ij}| = 1$ and $\gamma_{ij}t_{ij} = |t_{ij}|$ for each $i, j \in \mathbb{N}$. Since T is bounded, we have that $\sum_{j=1}^\infty |t_{ij}| \leq \|T\|$ for each i . Moreover,

since $T(c_0) \subset c_0$, we have $t_{ij} = e_i^* T e_j \rightarrow 0$ as $i \rightarrow \infty$. This also shows $\sum_{j=1}^m |t_{ij}| \rightarrow 0$ for each $m \in \mathbb{N}$ as $i \rightarrow \infty$.

Take an arbitrary $a = (a_n) \in \ell_\infty \setminus c_0(T)$. Then there exists an increasing sequence (i_k) of natural numbers such that $e_{i_k}^* T a \rightarrow \alpha \neq 0$. Removing finite number of elements from (i_k) if necessary, we have

$$|e_{i_k}^* T a - \alpha| < |\alpha|/2$$

for each k . Since $e_{i_k}^* T a = \sum_{j=1}^\infty t_{i_k, j} a_j$, it follows that

$$\|a\|_\infty \sum_{j=1}^\infty |t_{i_k, j}| \geq \left| \sum_{j=1}^\infty t_{i_k, j} a_j \right| = |e_{i_k}^* T a| > |\alpha|/2.$$

Hence, putting $M = |\alpha|/(2\|a\|_\infty) > 0$ yields $M < \sum_{j=1}^\infty |t_{i_k, j}| \leq \|T\|$ for each k .

If we put $n_1 = i_1$ then there exists an m_1 such that $\sum_{j=m_1+1}^\infty |t_{n_1, j}| < M/4$. In this case, we have

$$\sum_{j=1}^{m_1} |t_{n_1, j}| = \sum_{j=1}^\infty |t_{n_1, j}| - \sum_{j=m_1+1}^\infty |t_{n_1, j}| > M/2.$$

Now we assume that there exist strictly increasing sequences $(n_p)_{p=1}^q, (m_p)_{p=1}^q$ satisfying

- (i) $\sum_{j=1}^{m_{p-1}} |t_{n_p, j}| < M/2^{p+1}$;
- (ii) $\sum_{j=m_p+1}^\infty |t_{n_p, j}| < M/2^{p+1}$; and
- (iii) $\sum_{j=m_{p-1}+1}^{m_p} |t_{n_p, j}| > (1 - 1/2^p)M$

for each $p = 1, 2, \dots, q$, where $m_0 = 0$. Since $\sum_{j=1}^{m_q} |t_{ij}| \rightarrow 0$ as $i \rightarrow \infty$, there exists an $n_{q+1} \in (i_k)$ such that $\sum_{j=1}^{m_q} |t_{n_{q+1}, j}| < M/2^{q+2}$. For this n_{q+1} , there exists an $m_{q+1} \in \mathbb{N}$ with $m_{q+1} > m_q$ such that $\sum_{j=m_{q+1}+1}^\infty |t_{n_{q+1}, j}| < M/2^{q+2}$. It follows that

$$\begin{aligned} \sum_{j=m_q+1}^{m_{q+1}} |t_{n_{q+1}, j}| &= \sum_{j=1}^\infty |t_{n_{q+1}, j}| - \sum_{j=1}^{m_q} |t_{n_{q+1}, j}| - \sum_{j=m_{q+1}+1}^\infty |t_{n_{q+1}, j}| \\ &> (1 - 1/2^{q+1})M. \end{aligned}$$

Thus, by an induction, we have infinite sequences $(n_p)_{p=1}^\infty, (m_p)_{p=1}^\infty$ satisfying

- (i) $\sum_{j=1}^{m_{p-1}} |t_{n_p, j}| < M/2^{p+1}$;
- (ii) $\sum_{j=m_p+1}^\infty |t_{n_p, j}| < M/2^{p+1}$; and
- (iii) $\sum_{j=m_{p-1}+1}^{m_p} |t_{n_p, j}| > (1 - 1/2^p)M$

for each $p \in \mathbb{N}$. Let $N_p = \{m_{p-1} + 1, m_{p-1} + 2, \dots, m_p\}$ for each $p \in \mathbb{N}$, where $m_0 = 0$.

It is known that there exists a family $(A_\lambda)_{\lambda \in I}$ of subsets of \mathbb{N} with the following properties:

- (i) The index set I is uncountable.
- (ii) A_λ is an infinite set for each $\lambda \in I$.
- (iii) $A_\lambda \cap A_\mu$ is finite whenever $\lambda \neq \mu$.

See, for example, [12, Lemma 3.2.19]. For each $\lambda \in I$, define the bounded sequence $a^{(\lambda)} = (a_n^{(\lambda)})$ by

$$a_n^{(\lambda)} = \begin{cases} \gamma_{n,p,n} & (n \in N_p, p \in A_\lambda) \\ 0 & (n \in N_p, p \notin A_\lambda) \end{cases}.$$

We show that $a^{(\lambda)} \notin c(T)$. Indeed, we have

$$e_{n_p}^* Ta^{(\lambda)} = \sum_{j=1}^{m_p-1} t_{n_p j} a_j^{(\lambda)} + \sum_{j=m_p-1+1}^{m_p} t_{n_p j} a_j^{(\lambda)} + \sum_{j=m_p+1}^{\infty} t_{n_p j} a_j^{(\lambda)},$$

and hence, if $p \in A_\lambda$ then

$$|e_{n_p}^* Ta^{(\lambda)}| \geq \sum_{j=m_p-1+1}^{m_p} |t_{n_p j}| - \sum_{j=1}^{m_p-1} |t_{n_p j}| - \sum_{j=m_p+1}^{\infty} |t_{n_p j}| > (1 - 1/2^{p-1})M.$$

However, in the case of $p \notin A_\lambda$, one obtains

$$|e_{n_p}^* Ta^{(\lambda)}| \leq \sum_{j=1}^{m_p-1} |t_{n_p j}| + \sum_{j=m_p+1}^{\infty} |t_{n_p j}| < M/2^p.$$

Since A_λ and $\mathbb{N} \setminus A_\lambda$ are both infinite set, the sequence $Ta^{(\lambda)}$ cannot converge.

Next, we shall see that $a^{(\lambda)} - a^{(\mu)} \notin c(T)$ whenever $\lambda \neq \mu$. If $p \in A_\lambda \setminus A_\mu$, as in the preceding paragraph, we have

$$\operatorname{Re}[e_{n_p}^* (Ta^{(\lambda)} - Ta^{(\mu)})] > (1 - 3/2^p)M,$$

while

$$\operatorname{Re}[e_{n_p}^* (Ta^{(\lambda)} - Ta^{(\mu)})] < -(1 - 3/2^p)M,$$

for the case of $p \in A_\mu \setminus A_\lambda$. Remark that, in either case, one has

$$|\operatorname{Im}[e_{n_p}^* (Ta^{(\lambda)} - Ta^{(\mu)})]| < M/2^{p-1}.$$

From these estimations, we deduce that the sequence $Ta^{(\lambda)} - Ta^{(\mu)}$ is not Cauchy, and thus it does not converge.

We now consider the value of $\|(\sum_{j=1}^n \alpha_j a^{(\lambda_j)}) + c_0\|$, where $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ are mutually distinct elements of I . Since $A_{\lambda_i} \cap A_{\lambda_j}$ is finite whenever $i \neq j$, after removing finitely many coordinates, we can easily show that $\|(\sum_{j=1}^n \alpha_j a^{(\lambda_j)}) + c_0\| \leq \max_{1 \leq j \leq n} |\alpha_j|$.

Finally, let M be a closed subspace of ℓ_∞ with $c_0 \subset M \subset c(T)$. Then one has $a^{(\lambda)}, a^{(\lambda)} - a^{(\mu)} \notin M$ whenever $\lambda \neq \mu$. Let $\varphi \in (\ell_\infty/M)^*$. For each $\lambda \in I$, there exists a $\delta_\lambda \in \mathbb{C}$ such that $|\delta_\lambda| = 1$ and $\delta_\lambda \varphi(a^{(\lambda)} + M) = |\varphi(a^{(\lambda)} + M)|$. Take an arbitrary finite subset J of I . Then we obtain

$$\begin{aligned} \|\varphi\| &\geq \|\varphi\| \left\| \left(\sum_{\lambda \in J} \delta_\lambda a^{(\lambda)} \right) + c_0 \right\| \geq \|\varphi\| \left\| \left(\sum_{\lambda \in J} \delta_\lambda a^{(\lambda)} \right) + M \right\| \\ &\geq \left| \varphi \left(\left(\sum_{\lambda \in J} \delta_\lambda a^{(\lambda)} \right) + M \right) \right| = \sum_{\lambda \in J} |\varphi(a^{(\lambda)} + M)|, \end{aligned}$$

which implies that $I_{\varphi,n} = \{\lambda \in I : |\varphi(a^{(\lambda)} + M)| > 1/n\}$ is finite for each n . Hence $I_\varphi = \{\lambda \in I : \varphi(a^{(\lambda)} + M) \neq 0\} = \bigcup_n I_{\varphi,n}$ is countable. Now suppose that \mathcal{C} is a countable subset of $(\ell_\infty/M)^*$. Then it follows that

$$\{\lambda \in I : \varphi(a^{(\lambda)} + M) \neq 0 \text{ for some } \varphi \in \mathcal{C}\} = \bigcup_{\varphi \in \mathcal{C}} I_\varphi$$

is countable, and therefore \mathcal{C} cannot be total. This completes the proof. \square

REMARK 2.2. We remark that the preceding theorem is not true in general without the assumption on matrix representability. Indeed, there exists an operator $T \in B(\ell_\infty)$ which satisfies $c_0 \subset c_0(T) \subsetneq \ell_\infty$, but the conclusion of Theorem 2.1 does not hold. Indeed, let φ be a Banach limit on ℓ_∞ , and let $Ta = \varphi(a)\mathbf{1}$ for each $a \in \ell_\infty$. Then $T(c_0) = \{0\} \subset c_0$ and $T(\ell_\infty) = \mathbb{C}\mathbf{1} \not\subset c_0$. However the identity $c(T) = \ell_\infty$ holds. Hence the conclusion of Theorem 2.1 fails for this T .

The rest of this section is devoted to presenting some applications of Theorem 2.1. Recall that a sequence $a = (a_n) \in \ell_\infty$ is said to be *convergent in the sense of Cesàro mean of order 1* to α if the sequence $(n^{-1} \sum_{j=1}^n a_j)$ converges to α , and *almost convergent* to the *almost limit* α if $\varphi(a) = \alpha$ for each Banach limit φ on ℓ_∞ . It is well-known as Lorentz’s theorem [11] that $a = (a_n) \in \ell_\infty$ is almost convergent to α if and only if

$$\limsup_m \sup_{n \in \mathbb{N}} \left| \frac{1}{m} \sum_{j=1}^m a_{n+j-1} - \alpha \right| = 0.$$

The spaces of all bounded sequences convergent in the sense of Cesàro mean of order 1 is denoted by \tilde{c} . In [15], a similar sequence space (containing unbounded ones) was investigated by using the same symbol. The Banach spaces consisting of all almost convergent or almost null sequences are denoted by f and f_0 , respectively. We note that $c_0 \subset f_0 \subset f \subset \tilde{c}$ holds.

COROLLARY 2.3. *All the spaces \tilde{c}, f, f_0 are closed and uncomplemented in ℓ_∞ . Moreover, f_0 contains an isometric copy of ℓ_∞ . Consequently, \tilde{c}, f, f_0 are not prime.*

Proof. For each $a = (a_n) \in \ell_\infty$, let

$$Ta = \left(a_1, \frac{a_1 + a_2}{2}, \dots, \frac{a_1 + a_2 + \dots + a_n}{n}, \dots \right).$$

Then $T \in B(\ell_\infty)$ and admits a matrix representation (t_{ij}) , where

$$t_{ij} = \begin{cases} 1/i & (i \geq j) \\ 0 & (i < j) \end{cases}.$$

Moreover, we have $T(\mathbf{1}) = \mathbf{1} \notin c_0$. Hence, by Theorem 2.1, all closed subspaces M of ℓ_∞ satisfying $c_0 \subset M \subset c(T) = \tilde{c}$ are not complemented in ℓ_∞ .

For the fact that f_0 contains an isometric copy of ℓ_∞ , we refer the readers to Lorentz [11] (see also [3, Theorem 3.2]). The proof is complete. \square

COROLLARY 2.4. *Let d and d_0 be subspaces of ℓ_∞ given by*

$$\begin{aligned} c(\Delta) &= \{a = (a_n) \in \ell_\infty : (a_n - a_{n+1}) \text{ converges}\} \\ c_0(\Delta) &= \{a = (a_n) \in \ell_\infty : (a_n - a_{n+1}) \text{ converges to } 0\} \end{aligned}$$

Then $c(\Delta), c_0(\Delta)$ are closed and uncomplemented in ℓ_∞ . Moreover, $c_0(\Delta)$ contains an isomorphic copy of ℓ_∞ . Consequently, $c(\Delta), c_0(\Delta)$ are not prime.

Proof. Let Δ be a bounded linear operator on ℓ_∞ given by $\Delta a = (a_n - a_{n+1})$ for each $a = (a_n) \in \ell_\infty$. Then Δ admits a matrix representation (t_{ij}) , where $t_{ii} = 1$ and $t_{i(i+1)} = -1$ for each i , and $t_{ij} = 0$ for otherwise. We note that $\Delta(c) \subset c_0$ since each convergent sequence is Cauchy. Moreover, one has

$$\Delta(1, 0, 1, 0, \dots) = (1, -1, 1, -1, \dots) \notin c.$$

Hence Δ satisfies the assumption of Theorem 2.1. Now, it follows from $c_0 \subset c_0(\Delta) \subset c(\Delta)$ that $c(\Delta)$ and $c_0(\Delta)$ are closed and not complemented in ℓ_∞ .

We shall show that $c_0(\Delta)$ has an isomorphic copy of ℓ_∞ in it. Since $\sum_n 1/n = \infty$, we have an infinite sequence (m_k) such that $1/2 \leq \sum_{j=m_{k-1}+1}^{m_k} 1/j \leq 1$, where $m_0 = 0$. Put $M_k = \sum_{j=m_{k-1}+1}^{m_k} 1/j$ for each k . Then there exists an $n_k \in \mathbb{N}$ such that $M_k/n_k \leq 1/m_k$. Put $q_0 = 0$. Define p_k and q_k inductively by $p_k = q_{k-1} + m_k - m_{k-1}$ and $q_k = p_k + n_k$ for each $k \in \mathbb{N}$. It follows that $q_0 < p_1 < q_1 < p_2 < \dots$. Let $I_k = \{q_{k-1} + 1, q_{k-1} + 2, \dots, p_k\}$ and $J_k = \{p_k + 1, p_k + 2, \dots, q_k\}$ for each k . Then $|I_k| = m_k - m_{k-1}$, $|J_k| = n_k$ and

$$I_k \cup J_k = \{q_{k-1} + 1, q_{k-1} + 2, \dots, q_k\}.$$

Let $a = (a_n)$ be an element of ℓ_∞ given by

$$a_{q_{k-1}+l} = \sum_{j=m_{k-1}+1}^{m_{k-1}+l} 1/j$$

for each $1 \leq l \leq m_k - m_{k-1}$, and

$$a_{p_k+l} = a_{p_k} - M_k l/n_k = (1 - l/n_k)M_k$$

for each $1 \leq l \leq n_k$. In particular, one has that $a_{p_k} = M_k$ and $a_{q_k} = 0$. Moreover, if $k \in \mathbb{N}$, then we note that $a_{q_{k-1}+l} - a_{q_{k-1}+l+1} = 1/(m_{k-1} + l)$ for each $1 \leq l \leq m_k - m_{k-1}$, and $a_{p_k+l} - a_{p_k+l+1} = M_k/n_k \leq 1/m_k$ for each $1 \leq l \leq n_k$. One has $a_{q_k} - a_{q_{k+1}} = -1/(m_k + 1)$. These show that

$$\max_{n \in I_k \cup J_k} |a_n - a_{n+1}| = 1/(m_{k-1} + 1)$$

for each $k \in \mathbb{N}$.

Now, for each $b = (b_n) \in \ell_\infty$, we define $(\Phi b)_n = a_n b_k$ for each $n \in I_k \cup J_k$. By the preceding paragraph and the fact that $a_{q_{k+1}} = 1/(m_k + 1)$ for each k , we have $\Phi b \in c_0(\Delta)$. Moreover, since $1/2 \leq a_{p_k} = M_k \leq 1$ and $\|a\|_\infty \leq 1$, it follows that

$$\|b\|_\infty/2 \leq \|\Phi b\|_\infty \leq \|b\|_\infty.$$

This proves that $\Phi(\ell_\infty)$ is an isomorphic copy of ℓ_∞ in $c_0(\Delta)$. \square

We remark that the symbols $c(\Delta)$ and $c_0(\Delta)$ are used in [8] to denoting the spaces of all (possibly unbounded) difference convergent or difference null sequences.

3. A weak* closed subspace

In this section, we construct a weak* closed uncomplemented subspace of ℓ_∞ . For this, we refer some results on projection constants; see König [9] and Foucart and Skrzypek [6]. Let M be a closed subspace of a Banach space X . Then the relative projection constant of M in X is given by

$$\lambda(M, X) := \inf\{\|P\| : P \text{ is a bounded projection from } X \text{ onto } M\}.$$

For each $m, N \in \mathbb{N}$, we consider the value

$$\lambda(m, N) := \max\{\lambda(M, \ell_\infty^N) : \dim M = m\}.$$

THEOREM 3.1. *There exists an uncomplemented weak* closed subspace W of ℓ_∞ . Moreover, W contains an isometric copy of ℓ_∞ .*

Proof. Let (p_m) be the increasing sequence of prime numbers with $p_1 = 5$. As in [9] (or [6]), for each m , there exists an p_m -dimensional subspace M_m of $\ell_\infty^{p_m^2}$ such that $\lim_m \lambda(M_m, \ell_\infty^{p_m^2})/\sqrt{m} = 1$. Fix an $m \in \mathbb{N}$. Let $\{e_1^{(m)}, e_2^{(m)}, \dots, e_{p_m}^{(m)}\}$ be a basis for M_m . Then we have a basis $\{e_1^{(m)}, e_2^{(m)}, \dots, e_{p_m^2}^{(m)}\}$ for the whole space $\ell_\infty^{p_m^2}$. Let $f_j^{(m)}(\sum_{i=1}^{N_m} a_i e_i^{(m)}) = a_{m+j}$ for each $j = 1, 2, \dots, p_m^2 - p_m$. Then one has $M_m = \bigcap_{j=1}^{p_m^2 - p_m} \ker f_j^{(m)}$.

Now let $A_1 = \{1, 2, \dots, p_1^2\}$ and

$$A_m = \left\{ \left(\sum_{i=1}^{m-1} p_i^2 \right) + 1, \left(\sum_{i=1}^{m-1} p_i^2 \right) + 2, \dots, \sum_{i=1}^m p_i^2 \right\}.$$

Put $P_m(a) = a \cdot \chi_{A_m}$ for each m and for each $a \in \ell_\infty$. Then, for each m , there exists a natural identification $Q_m : \ell_\infty^{p_m^2} \rightarrow P_m(\ell_\infty)$. Define a subspace W of ℓ_∞ by the internal direct sum $\sum_{m=1}^\infty \oplus Q_m(M_m)$. In other words, $a \in W$ if and only if $P_m a \in Q_m(M_m)$ for each m . It follows that the space W can be written as

$$W = \bigcap \{ \ker(Q_m^{-1} P_m)^* f_j^{(m)} : m \in \mathbb{N}, 1 \leq j \leq p_m^2 - p_m \}.$$

Since each projection P_m is weak*-to-norm continuous, all the functional of the form $(Q_m^{-1} P_m)^* f_j^{(m)}$ are weakly* continuous, which proves that W is weak* closed.

Suppose that P is a bounded projection from ℓ_∞ onto W . Then the operator $Q_m^{-1} P_m P Q_m$ is a bounded projection from $\ell_\infty^{p_m^2}$ onto M_m . Indeed, we have $P_m P a \in Q_m(M_m) \subset W$ for each $a \in \ell_\infty$, which implies that $(P_m P)^2 = P_m P$. Hence one has

$$1 = \lim_m \frac{\lambda(M_m, \ell_\infty^{p_m^2})}{\sqrt{m}} \leq \limsup_m \frac{\|Q_m^{-1} P_m P Q_m\|}{\sqrt{m}} \leq \lim_m \frac{\|P\|}{\sqrt{m}} = 0,$$

a contradiction. Thus there is no bounded projection from ℓ_∞ onto W , that is, W is uncomplemented in ℓ_∞ .

Finally, take an arbitrary $x_m \in S_{Q_m(M_m)}$ for each m . Define $T : \ell_\infty \rightarrow W$ by $T(a_n) = w^*\text{-}\lim_m \sum_{i=1}^m a_i x_i$. It is routine to check that T is well-defined and isometric. The proof is complete. \square

On the other hand, it is known that if M is a weak* closed subspace of ℓ_∞ then $(\ell_\infty/M)^*$ always has a countable total subset. As a consequence, the property that $(\ell_\infty/M)^*$ has a countable total subset is necessary but not sufficient for assuring the complementarity of M in ℓ_∞ . Hence there exists a limit to determining the complementarity of subspaces of ℓ_∞ by using Whitley’s method while it still has interesting applications.

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