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ORTHONORMAL SEQUENCES AND TIME FREQUENCY LOCALIZATION RELATED TO THE RIEMANN–LIOUVILLE OPERATOR

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Abstract. For every real number $p>0$, we define the p-dispersion $\rho_{p,\nu_\alpha}(f)$ of a measurable function f on $[0,+\infty[\times\mathbb{R}$, where ν_α is some positive measure. We prove that for every orthonormal basis $(\varphi_{m,n})_{(m,n)\in\mathbb{N}^2}$ of $L^2(d\nu_\alpha)$, the sequences $\Big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\Big)_{(m,n)\in\mathbb{N}^2}$, $\Big(\rho_{p,\nu_\alpha}(\widetilde{\mathscr{F}}_\alpha(\varphi_{m,n}))\Big)_{(m,n)\in\mathbb{N}^2}$ can not be simultaneously bounded, where $\widetilde{\mathscr{F}}_\alpha$ is some Fourier transform. The main tool is a time frequency localization inequality for orthonormal sequences in $L^2(d\nu_\alpha)$.

On the other hand, we construct an orthonormal sequence $(\psi_{m,n})_{(m,n)\in\mathbb{N}^2}\subset L^2(d\nu_\alpha)$ such that the sequence $\Big(\rho_{p,\nu_\alpha}(\psi_{m,n})\rho_{p,\nu_\alpha}(\widetilde{\mathscr{F}}_\alpha(\psi_{m,n}))\Big)_{(m,n)\in\mathbb{N}^2}$ is bounded.

1. Introduction

The uncertainty principles play an important role in harmonic analysis. These principles state that a nonzero function f and its Fourier transform $\widehat{f}$ can not be simultaneously and sharply localized at the same time. Many mathematical formulations of this fact have been checked in the last decades [8, 10, 11, 16, 17, 25, 26]. In [30], Shapiro has studied the localization for an orthonormal sequence $(\varphi_k)_{k\in\mathbb{N}}$. He showed that if the means and the dispersions of the orthonormal sequence $(\varphi_k)_{k\in\mathbb{N}}$ and their Fourier transforms $(\widehat{\varphi}_k)_{k\in\mathbb{N}}$ are uniformly bounded, then $(\varphi_k)_{k\in\mathbb{N}}$ is finite. In [22], the authors gave a quantitative version of the precedent theorem, that is if $(\varphi_k)_{k\in\mathbb{N}}$ is an orthonormal sequence in $L^2(\mathbb{R})$, then for every $n\in\mathbb{N}$,

$$\sum_{k=0}^{n}\Big(||x\varphi_k||_2^2+||y\widehat{\varphi}_k||_2^2\Big)\geqslant\frac{(n+1)^2}{2\pi}.$$

Recently, in [24], the author obtains a quantitative multivariables version of Shapiro's theorem for generalized dispersion, in fact the author showed that if $(\varphi_k)_{k\in\mathbb{N}}$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$; then for every positive real number p and for every $n\in\mathbb{N}^*$

$$\sum_{k=1}^{n}\Big(||\,|x|^{\frac{p}{2}}\varphi_k\,||_2^2+||\,|y|^{\frac{p}{2}}\widehat{\varphi}_k\,||_2^2\Big)\geqslant C\,n^{1+\frac{p}{2d}}$$

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where C is a constant which does not depend on p. The author obtains also a multiplicative form of the above theorem by showing that if $(\varphi_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$, then for every positive real number p,

$$\sup_{k\in\mathbb{N}}\Big(||\,|x|^{\frac{p}{2}}\varphi_k\,||_2\,||\,|y|^{\frac{p}{2}}\widehat{\varphi}_k\,||_2\Big)=+\infty.$$

On the other hand, in [5], the authors have defined the Riemann-Liouville operator $\mathscr{R}_\alpha$; $\alpha\geqslant 0$, by

$$\mathscr{R}_\alpha(f)(r,x)=\begin{cases}\dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1}f(rs\sqrt{1-t^2},x+rt)(1-t^2)^{\alpha-\frac{1}{2}}\\ \qquad\qquad\qquad\times(1-s^2)^{\alpha-1}\,dt\,ds, & \text{if } \alpha>0,\\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1}f(r\sqrt{1-t^2},x+rt)\frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha=0;\end{cases}\tag{1.1}$$

where f is any continuous function on $\mathbb{R}^2$, even with respect to the first variable.

The dual operator ${}^t\mathscr{R}_\alpha$ is defined by

$${}^t\mathscr{R}_\alpha(g)(r,x)=\begin{cases}\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{2^\alpha\,\Gamma(\alpha+1)}\displaystyle\int_r^{+\infty}\int_{-\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}}g(u,x+v)\\ \qquad\qquad\qquad\times(u^2-v^2-r^2)^{\alpha-1}u\,du\,dv, & \text{if } \alpha>0,\\ \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}}g\big(\sqrt{r^2+(x-y)^2},y\big)dy, & \text{if } \alpha=0;\end{cases}\tag{1.2}$$

where g is any continuous function on $\mathbb{R}^2$, even with respect to the first variable and with compact support.

In particular, for $\alpha=0$ and by a change of variables, we get

$$\mathscr{R}_0(f)(r,x)=\frac{1}{2\pi}\int_0^{2\pi}f(r\cos\theta,x+r\sin\theta)d\theta.$$

This means that $\mathscr{R}_0(f)(r,x)$ is the mean value of f on the circle centered at $(0,x)$ and with radius r.

The mean operator $\mathscr{R}_0$ and its dual ${}^t\mathscr{R}_0$ play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [19, 20] or in the linearized inverse scattering problem in acoustics [15].

The operators $\mathscr{R}_\alpha$ and its dual ${}^t\mathscr{R}_\alpha$ have the same properties as the Radon transform [18], for this reason, $\mathscr{R}_\alpha$ is called sometimes the generalized Radon transform.

The Fourier transform $\mathscr{F}_\alpha$ associated with the operator $\mathscr{R}_\alpha$ is defined by

$$\begin{aligned}\forall(\lambda_0,\lambda)\in\Upsilon,\ \mathscr{F}_\alpha(f)(\lambda_0,\lambda)&=\int_0^\infty\int_{\mathbb{R}}f(r,x)\mathscr{R}_\alpha\big(\cos(\lambda_0\cdot)e^{-i\lambda\cdot}\big)(r,x)d\nu_\alpha(r,x)\\&=\int_0^\infty\int_{\mathbb{R}}f(r,x)\ j_\alpha(r\sqrt{\lambda_0^2+\lambda^2})e^{-i\lambda x}d\nu_\alpha(r,x),\end{aligned}$$

where Υ is the set given by

$$\Upsilon=\mathbb{R}^2\cup\big\{(i\lambda_0,\lambda);\ (\lambda_0,\lambda)\in\mathbb{R}^2;\ |\lambda_0|\leqslant|\lambda|\big\}.\tag{1.3}$$

$d\nu_\alpha(r,x)$ is the measure defined on $[0,+\infty[\times\mathbb{R}$ by

$$d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}dr}{2^\alpha\Gamma(\alpha+1)} \otimes \frac{dx}{\sqrt{2\pi}}. \tag{1.4}$$

j_α is the modified Bessel function that will be defined in the second section.

Many harmonic analysis results have been established for the Fourier transform $\mathscr{F}_\alpha$ [1, 4, 5, 6, 7, 28, 29]. Also, many uncertainty principles related to the Fourier transform $\mathscr{F}_\alpha$ have been proved [2, 3, 21, 26, 27].

Our investigation in this work is to prove a generalized quantitative version of the mean-dispersion Shapiro's theorem related to the Riemann-Liouville operator.

This paper is arranged as follows. In the second section, we collect some harmonic analysis results for the Riemann-Liouville operator $\mathscr{R}_\alpha$ and its connected Fourier transform $\mathscr{F}_\alpha$. The third section contains the main results of this work, we will prove a quantitative version of the mean-dispersion shapiro's theorem. Next, we establish a multiplicative form of this theorem.

2. The Riemann-Liouville transform

In this section, we recall some harmonic analysis results related to the convolution product and the Fourier transform associated with the Riemann-Liouville operator.

Let D and Ξ be the singular partial differential operators defined by

$$\begin{cases} D = \dfrac{\partial}{\partial x}; \\ \Xi = \dfrac{\partial^2}{\partial r^2} + \dfrac{2\alpha+1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial x^2}; \ (r,x) \in]0,+\infty[\times\mathbb{R}, \ \alpha \geqslant 0. \end{cases}$$

For all $(\lambda_0,\lambda) \in \mathbb{C}^2$, the system

$$\begin{cases} Du(r,x) = -i\lambda u(r,x); \\ \Xi u(r,x) = -\lambda_0^2 u(r,x); \\ u(0,0) = 1, \ \dfrac{\partial u}{\partial r}(0,x) = 0; \ \forall x \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\lambda_0,\lambda}$ given by

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R}, \quad \varphi_{\lambda_0,\lambda}(r,x) = j_\alpha\left(r\sqrt{\lambda_0^2+\lambda^2}\right) e^{-i\lambda x}, \tag{2.1}$$

where j_α is the modified Bessel function defined by

$$j_\alpha(z) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{z}{2}\right)^{2k},$$

and J_α is the Bessel function of first kind and index α [13, 14, 23, 35]. The modified Bessel function j_α has the integral representation

$$j_\alpha(z) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}\exp(-izt)dt. \tag{2.2}$$

Consequently, for every $k \in \mathbb{N}$ and $z \in \mathbb{C}$, we have

$$\left| j_\alpha^{(k)}(z) \right| \leqslant e^{|Im(z)|}. \tag{2.3}$$

PROPOSITION 2.1. *The eigenfunction $\varphi_{\lambda_0,\lambda}$ satisfies the following properties*

i. The function $\varphi_{\lambda_0,\lambda}$ is bounded on $\mathbb{R}^2$ if, and only if $(\lambda_0,\lambda) \in \Upsilon$, where Υ is the set given by the relation (1.3) and in this case,

$$\sup_{(r,x)\in \mathbb{R}^2} \left| \varphi_{\lambda_0,\lambda}(r,x) \right| = 1. \tag{2.4}$$

ii. The function $\varphi_{\lambda_0,\lambda}$ has the following Mehler integral representation

$$\varphi_{\lambda_0,\lambda}(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} \cos\left(\lambda_0 rs\sqrt{1-t^2}\right)\exp\left(-i\lambda(x+rt)\right) \\ \qquad\qquad \times(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}dtds; \text{ if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} \cos\left(r\lambda_0\sqrt{1-t^2}\right)\exp\left(-i\lambda(x+rt)\right) \\ \qquad\qquad \times\dfrac{dt}{\sqrt{1-t^2}}, \quad \text{if } \alpha=0. \end{cases} \tag{2.5}$$

REMARK 2.2. The Mehler integral representation (2.5) of the eigenfunction $\varphi_{\lambda_0,\lambda}$ allows us to define the integral transform $\mathscr{R}_\alpha$ by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2},x+rt)(1-t^2)^{\alpha-\frac{1}{2}} \\ \qquad\qquad \times(1-s^2)^{\alpha-1}\,dt\,ds, \text{ if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f(r\sqrt{1-t^2},x+rt)\dfrac{dt}{\sqrt{1-t^2}}, \qquad \text{if } \alpha=0; \end{cases} \tag{2.6}$$

where f is any continuous function on $\mathbb{R}^2$; even with respect to the first variable. Then, the relations (2.5) and (2.6) show that

$$\varphi_{\lambda_0,\,\lambda}(r,x) = \mathscr{R}_\alpha\left(\cos(\lambda_0\cdot)e^{-i\lambda\cdot}\right)(r,x), \tag{2.7}$$

which gives the mutual connection between the functions $\varphi_{\lambda_0,\,\lambda}$ and $\cos(\lambda_0\cdot)e^{-i\lambda\cdot}$.

For this reason, the operator $\mathscr{R}_\alpha$ is called the Riemann-Liouville transform associated with the operators D and Ξ.

The partial differential operators D and Ξ satisfy the intertwining properties with the Riemann-Liouville operator and its dual

$$^t\mathscr{R}_\alpha\Xi(f) = \frac{\partial^2}{\partial r^2}\,{}^t\mathscr{R}_\alpha(f), \qquad {}^t\mathscr{R}_\alpha D(f) = D\,{}^t\mathscr{R}_\alpha(f),$$

$$\Xi\mathscr{R}_\alpha(f) = \mathscr{R}_\alpha\frac{\partial^2}{\partial r^2}(f), \qquad D\mathscr{R}_\alpha(f) = \mathscr{R}_\alpha D(f),$$

where f is a sufficiently smooth function.

We denote by $L^p(d\nu_\alpha)$; $p\in[1,+\infty]$, the Lebesgue space formed by the measurable functions f on $[0,+\infty[\times\mathbb{R}$ such that $||f||_{p,\nu_\alpha}<+\infty$, with

$$||f||_{p,\nu_\alpha}=\begin{cases}\left(\int_0^{+\infty}\int_{\mathbb{R}}\big|f(r,x)\big|^p d\nu_\alpha(r,x)\right)^{\frac{1}{p}}, & \text{if } p\in[1,+\infty[,\\ \underset{(r,x)\in[0,+\infty[\times\mathbb{R}}{\text{ess sup}}\big|f(r,x)\big|, & \text{if } p=+\infty,\end{cases} \tag{2.8}$$

and $d\nu_\alpha$ is given by the relation (1.4).

$\langle\,.\,|\,.\,\rangle_{\nu_\alpha}$ the inner product on the Hilbert space $L^2(d\nu_\alpha)$ defined by

$$\langle f\mid g\rangle_{\nu_\alpha}=\int_0^\infty\int_{\mathbb{R}}f(r,x)\,\overline{g(r,x)}\,d\nu_\alpha(r,x).$$

$\mathscr{C}_{0,e}(\mathbb{R}^2)$ the space of continuous function on $\mathbb{R}^2$, even with respect to the first variable such that

$$\lim_{r^2+x^2\to+\infty}f(r,x)=0,$$

the space $\mathscr{C}_{0,e}(\mathbb{R}^2)$ is equipped with the norm

$$||f||_{\infty,\nu_\alpha}=\sup_{(r,x)\in[0,+\infty[\times\mathbb{R}}|f(r,x)|.$$

To define the translation operator associated with the Riemann-Liouville transform, we use the product formula for the eigenfunction $\varphi_{\lambda_0,\lambda}$, that is for $(r,x),(s,y)\in[0,+\infty[\times\mathbb{R}$,

$$\varphi_{\lambda_0,\lambda}(r,x)\varphi_{\lambda_0,\lambda}(s,y)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi\varphi_{\lambda_0,\lambda}\left(\sqrt{r^2+s^2+2rs\cos\theta},x+y\right)\sin^{2\alpha}(\theta)d\theta.$$

DEFINITION 2.3. i) For every $(r,x)\in[0,+\infty[\times\mathbb{R}$, the translation operator $\tau_{(r,x)}$ associated with the Riemann-Liouville transform is defined on $L^p(d\nu_\alpha)$; $p\in[1,+\infty]$, by

$$\tau_{(r,x)}f(s,y)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta},x+y\right)\sin^{2\alpha}(\theta)d\theta. \tag{2.9}$$

ii) The convolution product of $f,g\in L^1(d\nu_\alpha)$ is defined for every $(r,x)\in[0,+\infty[\times\mathbb{R}$, by

$$f*g(r,x)=\int_o^{+\infty}\int_{\mathbb{R}}\tau_{(r,-x)}(\check{f})(s,y)g(s,y)d\nu_\alpha(s,y), \tag{2.10}$$

where $\check{f}(s,y)=f(s,-y)$.

The set $[0,+\infty[\times\mathbb{R}$ equipped with the convolution product $*$ is an hypergroup in the sense of [9].

PROPOSITION 2.4.

i. *For every* $f \in L^p(d\nu_\alpha)$; $p \in [1,+\infty]$, *and for every* $(r,x) \in [0,+\infty[\times\mathbb{R}$, *the function* $\tau_{(r,x)}(f)$ *belongs to* $L^p(d\nu_\alpha)$ *and we have*

$$\left|\left|\tau_{(r,x)}(f)\right|\right|_{p,\nu_\alpha} \leqslant ||f||_{p,\nu_\alpha}. \tag{2.11}$$

ii. *For every* $f \in L^1(d\nu_\alpha)$ *and* $(r,x) \in [0,+\infty[\times\mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}} \tau_{(r,x)}(f)(s,y)d\nu_\alpha(s,y) = \int_0^\infty \int_{\mathbb{R}} f(s,y)d\nu_\alpha(s,y). \tag{2.12}$$

iii. *For every* $f \in L^p(d\nu_\alpha)$; $p \in [1,+\infty[$, *we have*

$$\lim_{(r,x)\to(0,0)} \left|\left|\tau_{(r,x)}(f) - f\right|\right|_{p,\nu_\alpha} = 0. \tag{2.13}$$

iv. *For every* $f \in \mathscr{C}_{0,e}(\mathbb{R}^2)$ *and every* $(r,x) \in \mathbb{R}^2$, *the function* $\tau_{(r,x)}(f)$ *belongs to* $\mathscr{C}_{0,e}(\mathbb{R}^2)$ *and*

$$\lim_{(r,x)\to(0,0)} \left|\left|\tau_{(r,x)}(f) - f\right|\right|_{\infty,\nu_\alpha} = 0. \tag{2.14}$$

v. *Let* φ *be a nonnegative measurable function on* $\mathbb{R}\times\mathbb{R}$, *even with respect to the first variable, such that*

$$\int_0^{+\infty} \int_{\mathbb{R}} \varphi(r,x)d\nu_\alpha(r,x) = 1.$$

Then the family $(\varphi_{(a,b)})_{(a,b)\in (\mathbb{R}_+^*)^2}$ *defined by*

$$\forall (r,x) \in \mathbb{R}\times\mathbb{R},\ \varphi_{(a,b)}(r,x) = \frac{1}{a^{2\alpha+2}b}\varphi\left(\frac{r}{a},\frac{x}{b}\right)$$

is an approximation of the identity in $L^p(d\nu_\alpha)$; $p \in [1,+\infty[$, *that is for every* $f \in L^p(d\nu_\alpha)$, *we have*

$$\lim_{(a,b)\to(0^+,0^+)} ||f*\varphi_{(a,b)} - f||_{p,\nu_\alpha} = 0. \tag{2.15}$$

vi. *For every* $f \in \mathscr{C}_{0,e}(\mathbb{R}^2)$,

$$\lim_{(a,b)\to(0^+,0^+)} ||f*\varphi_{(a,b)} - f||_{\infty,\nu_\alpha} = 0. \tag{2.16}$$

vii. *If* $1 \leqslant p,q,r \leqslant +\infty$ *are such that* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ *and if* $f \in L^p(d\nu_\alpha)$, $g \in L^q(d\nu_\alpha)$, *then the function* $f*g$ *belongs to* $L^r(d\nu_\alpha)$, *and we have the Young's inequality*

$$||f*g||_{r,\nu_\alpha} \leqslant ||f||_{p,\nu_\alpha}||g||_{q,\nu_\alpha}. \tag{2.17}$$

In the following, we need the notations Υ_+ is the subset of Υ given by

$$\Upsilon_+ = \mathbb{R}_+ \times \mathbb{R} \cup \big\{(it,x);\ (t,x) \in \mathbb{R}^2;\ 0 \leqslant t \leqslant |x|\big\}.$$

$\mathscr{B}_{\Upsilon_+}$ is the σ-algebra defined on Υ_+ by

$$\mathscr{B}_{\Upsilon_+} = \big\{\theta^{-1}(B),\ B \in \mathscr{B}_{\rm or}\big([0,+\infty[\times\mathbb{R}\big)\big\},$$

where θ is the bijective function defined on the set Υ_+ by

$$\theta(\lambda_0,\lambda) = \big(\sqrt{\lambda_0^2+\lambda^2},\lambda\big), \tag{2.18}$$

and $\mathscr{B}_{\rm or}\big([0,+\infty[\times\mathbb{R}\big)$ is the usual Borel σ-algebra on $[0,+\infty[\times\mathbb{R}$.

$d\gamma_\alpha$ is the measure defined on $\mathscr{B}_{\Upsilon_+}$ by

$$\forall A \in \mathscr{B}_{\Upsilon_+},\ \gamma_\alpha(A) = \nu_\alpha\big(\theta(A)\big).$$

$L^p(d\gamma_\alpha)$; $p \in [1,+\infty]$, the Lebesgue space consisting of measurable function g on Υ_+ such that

$$\|g\|_{p,\gamma_\alpha} < +\infty.$$

$\langle\ .\ |\ .\ \rangle_{\gamma_\alpha}$ the inner product on the Hilbert space $L^2(d\gamma_\alpha)$ given by

$$\langle\, f \mid g\, \rangle_{\gamma_\alpha} = \int\!\!\int_{\Upsilon_+} f(\lambda_0,\lambda)\,\overline{g(\lambda_0,\lambda)}\,d\gamma_\alpha(\lambda_0,\lambda).$$

PROPOSITION 2.5.

i. *For all nonnegative measurable function g on Υ_+, we have*

$$\begin{aligned}&\int\!\!\int_{\Upsilon_+} g(\lambda_0,\lambda)d\gamma_\alpha(\lambda_0,\lambda)\\ &= \frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\Big(\int_0^{+\infty}\int_{\mathbb{R}} g(\lambda_0,\lambda)(\lambda_0^2+\lambda^2)^\alpha\lambda_0\,d\lambda_0\,d\lambda\\ &\quad+\int_{\mathbb{R}}\int_0^{|\lambda|} g(i\lambda_0,\lambda)(\lambda^2-\lambda_0^2)^\alpha\lambda_0\,d\lambda_0\,d\lambda\Big).\end{aligned}$$

ii. *For all nonnegative measurable function f on $[0,+\infty[\times\mathbb{R}$ (respectively integrable on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $d\nu_\alpha$), $f\circ\theta$ is a nonnegative measurable function on Υ_+ (respectively integrable on Υ_+ with respect to the measure $d\gamma_\alpha$) and we have*

$$\int\!\!\int_{\Upsilon_+} (f\circ\theta)(\lambda_0,\lambda)d\gamma_\alpha(\lambda_0,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)d\nu_\alpha(r,x). \tag{2.19}$$

Now, using the eigenfunction $\varphi_{\lambda_0,\lambda}$ given by the relation (2.1), we can define the Fourier transform.

DEFINITION 2.6. The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$ by

$$\forall(\lambda_0,\lambda)\in\Upsilon,\qquad \mathscr{F}_\alpha(f)(\lambda_0,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\varphi_{\lambda_0,\lambda}(r,x)d\nu_\alpha(r,x).$$

PROPOSITION 2.7.

i. For every $f\in L^1(d\nu_\alpha)$, the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^\infty(d\gamma_\alpha)$ and we have

$$\big|\big|\mathscr{F}_\alpha(f)\big|\big|_{\infty,\gamma_\alpha}\leqslant ||f||_{1,\nu_\alpha}. \tag{2.20}$$

ii. Let $f\in L^1(d\nu_\alpha)$. For every $(r,x)\in[0,+\infty[\times\mathbb{R}$, we have

$$\forall(\lambda_0,\lambda)\in\Upsilon,\qquad \mathscr{F}_\alpha\big(\tau_{(r,x)}(f)\big)(\lambda_0,\lambda)=\overline{\varphi_{\lambda_0,\lambda}(r,x)}\mathscr{F}_\alpha(f)(\lambda_0,\lambda).$$

iii. For $f,g\in L^1(d\nu_\alpha)$, we have

$$\forall(\lambda_0,\lambda)\in\Upsilon,\quad \mathscr{F}_\alpha(f*g)(\lambda_0,\lambda)=\mathscr{F}_\alpha(f)(\lambda_0,\lambda)\mathscr{F}_\alpha(g)(\lambda_0,\lambda). \tag{2.21}$$

vi. For $f\in L^1(d\nu_\alpha)$, we have

$$\forall(\lambda_0,\lambda)\in\Upsilon,\qquad \mathscr{F}_\alpha(f)(\lambda_0,\lambda)=\widetilde{\mathscr{F}_\alpha}(f)\circ\theta(\lambda_0,\lambda), \tag{2.22}$$

where for every $(\lambda_0,\lambda)\in\mathbb{R}^2$,

$$\widetilde{\mathscr{F}_\alpha}(f)(\lambda_0,\lambda)=\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)j_\alpha(r\lambda_0)\exp(-i\lambda x)d\nu_\alpha(r,x), \tag{2.23}$$

and θ is the function defined by the relation (2.18).

Also, the Fourier transform $\mathscr{F}_\alpha$ satisfies the following properties

THEOREM 2.8.

i. Let $f\in L^1(d\nu_\alpha)$ such that the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^1(d\gamma_\alpha)$, then we have the following inversion formula for $\mathscr{F}_\alpha$, for almost every $(r,x)\in[0,+\infty[\times\mathbb{R}$,

$$\begin{aligned} f(r,x)&=\int\int_{\Upsilon_+}\mathscr{F}_\alpha(f)(\lambda_0,\lambda)\overline{\varphi_{\lambda_0,\lambda}(r,x)}d\gamma_\alpha(\lambda_0,\lambda)\\ &=\int_0^{\infty}\int_{\mathbb{R}}\widetilde{\mathscr{F}_\alpha}(f)(\lambda_0,\lambda)\ j_\alpha(r\lambda_0)\ e^{i\lambda x}\ d\nu_\alpha(\lambda_0,\lambda). \end{aligned} \tag{2.24}$$

ii. (Plancherel theorem) *The Fourier transform $\mathscr{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$ and for every $f\in L^2(d\nu_\alpha)$,*

$$||\mathscr{F}_\alpha(f)||_{2,\gamma_\alpha}=||f||_{2,\nu_\alpha}. \tag{2.25}$$

In particular, we have the Parseval equality; for all $f,g\in L^2(d\nu_\alpha)$,

$$\langle\ f\mid g\ \rangle_{\nu_\alpha}=\langle\ \mathscr{F}_\alpha(f)\mid\mathscr{F}_\alpha(g)\ \rangle_{\gamma_\alpha}. \tag{2.26}$$

Using the relations (2.20), (2.25) and the Riesz-Thorin theorem's [31, 32], we deduce that for every $f \in L^p(d\nu_\alpha)$; $p \in [1,2]$, the function $\mathscr{F}_\alpha(f)$ lies in $L^{p'}(d\gamma_\alpha)$; $p' = \dfrac{p}{p-1}$, and we have

$$\big|\big|\mathscr{F}_\alpha(f)\big|\big|_{p',\gamma_\alpha} \leqslant ||f||_{p,\nu_\alpha}. \tag{2.27}$$

We denote by $\mathscr{S}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives, even with respect to the first variable.

The space $\mathscr{S}_e(\mathbb{R}^2)$ is endowed with the topology generated by the family of norms

$$\rho_m(\varphi) = \sup_{\substack{(r,x)\in\ [0,+\infty[\times\mathbb{R} \\ k+|\beta|\leqslant m}} \big(1+r^2+x^2\big)^k \big|D^\beta(\varphi)(r,x)\big|. \tag{2.28}$$

$\mathscr{D}_e(\mathbb{R}^2)$ the subspace of $\mathscr{S}_e(\mathbb{R}^2)$ formed by the functions with compact support.

From [33, 34], the transform $\widetilde{\mathscr{F}_\alpha}$ given by the relation (2.23) is a topological isomorphisme from $\mathscr{S}_e(\mathbb{R}^2)$ onto itself and we have

$$\widetilde{\mathscr{F}_\alpha}^{-1}(f)(r,x) = \int_0^\infty \int_{\mathbb{R}} f(\lambda_0,\lambda)\ j_\alpha(r\lambda_0)\ e^{i\lambda x} d\nu_\alpha(\lambda_0,\lambda) = \widetilde{\mathscr{F}_\alpha}(\check{f})(r,x).$$

3. Main results

In this section we shall prove the dispersion principle and one multiplicative form related to the Riemann-Liouville operator. For this, we need some intermediate results.

DEFINITION 3.1. Let p be a positive real number.

i. For every measurable function f on $[0,+\infty[\times\mathbb{R}$, the p-dispersion of f with respect to the measure $d\nu_\alpha$ is defined by

$$\rho_{p,\nu_\alpha}(f) = \Big(\int_0^\infty \int_{\mathbb{R}} |(r,x)|^p\ |f(r,x)|^2\ d\nu_\alpha(r,x)\Big)^{\frac{1}{p}}.$$

ii. For every measurable function g on Υ_+, the p-dispersion of g with respect to the measure $d\gamma_\alpha$ is defined by

$$\rho_{p,\gamma_\alpha}(g) = \Big(\int\int_{\Upsilon_+} |\theta(\lambda_0,\lambda)|^p\ |g(\lambda_0,\lambda)|^2\ d\gamma_\alpha(\lambda_0,\lambda)\Big)^{\frac{1}{p}}.$$

DEFINITION 3.2. Let ε be a positive real number and let f be a square integrable function on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $d\nu_\alpha$.

i. We say that f is ε-concentrated in the ball $B_\rho^+ = \{(r,x) \in\ [0,+\infty[\times\mathbb{R};\ r^2+x^2 \leqslant \rho^2\}$ if

$$\Big(\int\int_{(B_\rho^+)^c} |f(r,x)|^2\ d\nu_\alpha(r,x)\Big)^{\frac{1}{2}} \leqslant \varepsilon\ ||f||_{2,\nu_\alpha}.$$

ii. We say that f is ε-bandlimited in the ball $\widetilde{B}^+_\rho = \{(\lambda_0,\lambda) \in \Upsilon_+;\ |\theta(\lambda_0,\lambda)|^2 = \lambda_0^2 + 2\lambda^2 \leqslant \rho^2\}$ if

$$\Big(\int\int_{(\widetilde{B}^+_\rho)^c} \left|\mathscr{F}_\alpha(f)(\lambda_0,\lambda)\right|^2 d\gamma_\alpha(\lambda_0,\lambda)\Big)^{\frac{1}{2}} \leqslant \varepsilon\, ||f||_{2,\nu_\alpha}.$$

Let S be a measurable subset of $[0,+\infty[\times\mathbb{R}$ and let Σ be a measurable subset of Υ_+ such that

$$\nu_\alpha(S) < +\infty \text{ and } \gamma_\alpha(\Sigma) < +\infty.$$

We denote by P_S and P_Σ the bounded self adjoint operators defined on $L^2(d\nu_\alpha)$ respectively by $P_S(f) = \mathbf{1}_S.f$ and $P_\Sigma(f) = \mathscr{F}_\alpha^{-1}\big(\mathbf{1}_\Sigma\mathscr{F}_\alpha(f)\big)$.

We have the following interesting result

THEOREM 3.3. *The operators P_SP_Σ and $P_\Sigma P_S$ are Hilbert-Schmidt operators such that*

$$||P_SP_\Sigma||_{HS} \leqslant \sqrt{\nu_\alpha(S)\gamma_\alpha(\Sigma)} \qquad \textit{and} \qquad ||P_\Sigma P_S||_{HS} \leqslant \sqrt{\nu_\alpha(S)\gamma_\alpha(\Sigma)},$$

where $||.||_{HS}$ denotes the Hilbert-Schmidt norm.

Proof. Since $\nu_\alpha(S) < +\infty$ and $\gamma_\alpha(\Sigma) < +\infty$, then for every $f \in L^2(d\nu_\alpha)$, $\mathbf{1}_S.f$ belongs to $L^1(d\nu_\alpha)\cap L^2(d\nu_\alpha)$ and for every $g \in L^2(d\gamma_\alpha)$, $\mathbf{1}_\Sigma.g$ belongs to $L^1(d\gamma_\alpha)\cap L^2(d\gamma_\alpha)$. Consequently, for every $f \in L^2(d\nu_\alpha)$,

$$\begin{aligned}
P_\Sigma P_S(f)(r,x) &= \mathscr{F}_\alpha^{-1}\big(\mathbf{1}_\Sigma\mathscr{F}_\alpha(\mathbf{1}_S.f)\big)(r,x)\\
&= \int\int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0,\lambda)\ \mathscr{F}_\alpha(\mathbf{1}_S.f)(\lambda_0,\lambda)\ \overline{\varphi_{\lambda_0,\lambda}(r,x)}\ d\gamma_\alpha(\lambda_0,\lambda)\\
&= \int\int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0,\lambda)\overline{\varphi_{\lambda_0,\lambda}(r,x)}\\
&\quad \times\Big(\int_0^\infty\int_{\mathbb{R}} \mathbf{1}_S(t,y)\ f(t,y)\ \varphi_{\lambda_0,\lambda}(t,y)\ d\nu_\alpha(t,y)\Big)d\gamma_\alpha(\lambda_0,\lambda).
\end{aligned}$$

Applying Fubini's theorem, we get

$$\begin{aligned}
P_\Sigma P_S(f)(r,x) &= \int_0^\infty\int_{\mathbb{R}} \mathbf{1}_S(t,y)\ f(t,y)\\
&\quad\times\Big(\int\int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0,\lambda)\varphi_{\lambda_0,\lambda}(t,y)\overline{\varphi_{\lambda_0,\lambda}(r,x)}d\gamma_\alpha(\lambda_0,\lambda)\Big)d\nu_\alpha(t,y)\\
&= \int_0^\infty\int_{\mathbb{R}} f(t,y)\ K\big((r,x),(t,y)\big)d\nu_\alpha(t,y), \qquad (3.1)
\end{aligned}$$

where K is the kernel given by

$$\begin{aligned}
K\big((r,x),(t,y)\big) &= \mathbf{1}_S(t,y)\Big(\int\int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0,\lambda)\varphi_{\lambda_0,\lambda}(t,y)\overline{\varphi_{\lambda_0,\lambda}(r,x)}d\gamma_\alpha(\lambda_0,\lambda)\Big)\\
&= \mathbf{1}_S(t,y)\mathscr{F}_\alpha^{-1}\big(\mathbf{1}_\Sigma\varphi_{.\,,\,.}(t,y)\big)(r,x).
\end{aligned}$$

Using the Plancherel theorem for $\mathscr{F}_\alpha$, Fubini's theorem and the relation (2.4), we get

$$||K||^2_{2,\nu_\alpha\otimes\nu_\alpha} = \int_0^\infty\int_{\mathbb{R}} \mathbf{1}_S(t,y)\Big(\int\int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0,\lambda)|\varphi_{\lambda_0,\lambda}(t,y)|^2 d\gamma_\alpha(\lambda_0,\lambda)\Big)d\nu_\alpha(t,y)$$
$$\leqslant \nu_\alpha(S)\gamma_\alpha(\Sigma). \tag{3.2}$$

The relations (3.1) and (3.2) show that $P_\Sigma P_S$ is an Hilbert Schmidt operator and that

$$||P_\Sigma P_S||_{HS} = ||K||_{2,\nu_\alpha\otimes\nu_\alpha} \leqslant \sqrt{\nu_\alpha(S)\gamma_\alpha(\Sigma)}. \tag{3.3}$$

As the same way, for every $f \in L^1(d\nu_\alpha)\cap L^2(d\nu_\alpha)$ and for every $(r,x) \in [0,+\infty[\times\mathbb{R}$,

$$\begin{aligned} P_S P_\Sigma(f)(r,x) &= \mathbf{1}_S(r,x)\mathscr{F}_\alpha^{-1}\big(\mathbf{1}_\Sigma\mathscr{F}_\alpha(f)\big)(r,x) \\ &= \mathbf{1}_S(r,x)\int\int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0,\lambda)\mathscr{F}_\alpha(f)(\lambda_0,\lambda)\,\overline{\varphi_{\lambda_0,\lambda}(r,x)}d\gamma_\alpha(\lambda_0,\lambda) \\ &= \mathbf{1}_S(r,x)\int\int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0,\lambda)\Big(\int_0^\infty\int_{\mathbb{R}} f(s,y)\varphi_{\lambda_0,\lambda}(s,y)d\nu_\alpha(s,y)\Big) \\ &\quad\times\overline{\varphi_{\lambda_0,\lambda}(r,x)}d\gamma_\alpha(\lambda_0,\lambda). \end{aligned}$$

Applying Fubini's theorem, we have

$$\begin{aligned} P_S P_\Sigma(f)(r,x) &= \mathbf{1}_S(r,x)\int_0^\infty\int_{\mathbb{R}} f(s,y)\Big(\int\int_{\Upsilon_+} \mathbf{1}_\Sigma(\lambda_0,\lambda)\varphi_{\lambda_0,\lambda}(r,-x) \\ &\qquad\times\overline{\varphi_{\lambda_0,\lambda}(s,-y)}d\gamma_\alpha(\lambda_0,\lambda)\Big)d\nu_\alpha(s,y) \\ &= \int_0^\infty\int_{\mathbb{R}} f(s,y)H\big((r,x),(s,y)\big)d\nu_\alpha(s,y), \end{aligned}$$

where H is the kernel given by

$$H\big((r,x),(s,y)\big) = \mathbf{1}_S(r,x)\mathscr{F}_\alpha^{-1}\big(\mathbf{1}_\Sigma\varphi_{.,.}(r,-x)\big)(s,-y).$$

Applying again Fubini-Tonnelli theorem and the Plancherel theorem for $\mathscr{F}_\alpha$, we get

$$\begin{aligned} &\int_{([0,+\infty\times\mathbb{R}[)^2} \Big|H\big((r,x),(s,y)\big)\Big|^2 d\nu_\alpha(r,x)d\nu_\alpha(s,y) \\ &= \int_0^\infty\int_{\mathbb{R}} \mathbf{1}_S(r,x)\Big(\int_0^\infty\int_{\mathbb{R}} \Big|\mathscr{F}_\alpha^{-1}\big(\mathbf{1}_\Sigma\varphi_{.,.}(r,-x)\big)(s,y)\Big|^2 d\nu_\alpha(s,y)\Big)d\nu_\alpha(r,x) \\ &= \int_0^\infty\int_{\mathbb{R}} \mathbf{1}_S(r,x)\Big(\int\int_{\Upsilon_+} \big|\mathbf{1}_\Sigma(\lambda_0,\lambda)\varphi_{\lambda_0,\lambda}(r,-x)\big|^2 d\gamma_\alpha(\lambda_0,\lambda)\Big)d\nu_\alpha(r,x) \\ &\leqslant \nu_\alpha(S)\gamma_\alpha(\Sigma). \end{aligned} \tag{3.4}$$

The last inequality shows that the operator $P_S P_\Sigma$ is an Hilbert-Schmidt operator and that

$$||P_S P_\Sigma||_{HS} = ||H||_{2,\nu_\alpha\otimes\nu_\alpha} \leqslant \sqrt{\nu_\alpha(S)\gamma_\alpha(\Sigma)}. \quad \square$$

THEOREM 3.4. (Time frequency localization) *Let $S\subset[0,+\infty[\times\mathbb{R}$; $\Sigma\subset\Upsilon_+$ such that $\nu_\alpha(S)<+\infty$ and $\gamma_\alpha(\Sigma)<+\infty$. Let $\mathscr{K}$ be a finite subset of $\mathbb{N}^2$ and let $(\varphi_{m,n})_{(m,n)\in\mathscr{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Then*

$$\sum_{(m,n)\in\mathscr{K}}\big(1-\frac{3}{2}a_{m,n}(S)-\frac{3}{2}b_{m,n}(\Sigma)\big)\leqslant\nu_\alpha(S)\gamma_\alpha(\Sigma), \tag{3.5}$$

where

$$a_{m,n}(S)=||\mathbf{1}_{S^c}\varphi_{m,n}||_{2,\nu_\alpha}\text{ and }b_{m,n}(\Sigma)=||\mathbf{1}_{\Sigma^c}\mathscr{F}_\alpha(\varphi_{m,n})||_{2,\gamma_\alpha}. \tag{3.6}$$

Proof. For every $(m,n)\in\mathbb{N}^2$; we put

$$e^\alpha_{m,n}(r,x)=\Big(\frac{2^{\alpha+1}m!\,\Gamma(\alpha+1)}{2^{n-\frac{1}{2}}n!\Gamma(m+\alpha+1)}\Big)^{\frac{1}{2}}e^{-\frac{r^2+x^2}{2}}L^\alpha_m(r^2)\,H_n(x)$$

where $(L^\alpha_m)_{m\in\mathbb{N}}$ are the Laguerre polynomials and $(H_n)_{n\in\mathbb{N}}$ are the Hermite polynomials. Then, $(e^\alpha_{m,n})_{(m,n)\in\mathbb{N}^2}$ is an Hilbert basis of $L^2(d\nu_\alpha)$ [27]. Moreover, the family $(\mathscr{E}^\alpha_{m,n})_{(m,n)\in\mathbb{N}^2}$ defined by

$$\mathscr{E}^\alpha_{m,n}(\lambda_0,\lambda)=e^\alpha_{m,n}\circ\theta(\lambda_0,\lambda)$$

is an Hilbert basis of $L^2(d\gamma_\alpha)$ such that

$$\mathscr{F}_\alpha(e^\alpha_{m,n})=(-i)^{2m+n}\mathscr{E}^\alpha_{m,n}. \tag{3.7}$$

On the other hand, for every bounded operator T on $L^2(d\nu_\alpha)$, we denote by T^* the adjoint operator of T defined by

$$\langle T(f)\mid g\rangle_{\nu_\alpha}=\langle f\mid T^*(g)\rangle_{\nu_\alpha}\ ;\ f,g\in L^2(d\nu_\alpha).$$

Then, the operators P_S and P_Σ are self adjoint and satisfy $P_S^2=P_S$, $P_\Sigma^2=P_\Sigma$. Let ϕ be the self adjoint operator defined by $\phi=(P_\Sigma P_S)^*(P_\Sigma P_S)$, then ϕ can be written $\phi=P_SP_\Sigma P_S$ and ϕ is an operator with trace such that

$$\begin{aligned}tr(\phi)&=\sum_{(m,n)\in\mathbb{N}^2}||P_\Sigma P_S(e^\alpha_{m,n})||^2_{2,\nu_\alpha}\\&=\sum_{(m,n)\in\mathbb{N}^2}\langle\phi(e^\alpha_{m,n})|e^\alpha_{m,n}\rangle_{\nu_\alpha}\\&=||P_\Sigma P_S||^2_{HS}.\end{aligned}$$

Applying Theorem 3.3, we get

$$\sum_{(m,n)\in\mathbb{N}^2}\langle\phi(e^\alpha_{m,n})|e^\alpha_{m,n}\rangle_{\nu_\alpha}\leqslant\nu_\alpha(S)\gamma_\alpha(\Sigma).$$

Now, let $(\varphi_{m,n})_{(m,n)\in\mathscr{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, then $(\varphi_{m,n})_{(m,n)\in\mathscr{K}}$ can be completed to an Hilbert basis of $L^2(d\nu_\alpha)$ denoted by $(\varphi_{m,n})_{(m,n)\in\mathbb{N}^2}$. So,

$$\begin{aligned}\sum_{(m,n)\in\mathbb{N}^2}||P_\Sigma P_S(\varphi_{m,n})||^2_{2,\nu_\alpha} &= \sum_{(m,n)\in\mathbb{N}^2}||P_\Sigma P_S(e^\alpha_{m,n})||^2_{2,\nu_\alpha}\\ &= tr(\phi)\leqslant \nu_\alpha(S)\gamma_\alpha(\Sigma).\end{aligned}$$

In particular,

$$\begin{aligned}\sum_{(m,n)\in\mathscr{K}}||P_\Sigma P_S(\varphi_{m,n})||^2_{2,\nu_\alpha} &= \sum_{(m,n)\in\mathbb{N}^2}\langle\phi(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha}\\ &\leqslant \nu_\alpha(S)\gamma_\alpha(\Sigma).\end{aligned}\tag{3.8}$$

On the other hand, for every $(m,n)\in\mathbb{N}^2$,

$$\begin{aligned}\langle\phi(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha} &= \langle P_SP_\Sigma P_S(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha} = \langle P_\Sigma P_S(\varphi_{m,n})|P_S(\varphi_{m,n})\rangle_{\nu_\alpha}\\ &= \langle P_\Sigma P_S(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha} - \langle P_\Sigma P_S(\varphi_{m,n})|\varphi_{m,n}-P_S(\varphi_{m,n})\rangle_{\nu_\alpha},\end{aligned}$$

but,

$$\begin{aligned}\langle P_\Sigma P_S(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha} &= \langle P_S(\varphi_{m,n})|P_\Sigma(\varphi_{m,n})\rangle_{\nu_\alpha}\\ &= \langle P_S(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha} - \langle P_S(\varphi_{m,n})|\varphi_{m,n}-P_\Sigma(\varphi_{m,n})\rangle_{\nu_\alpha}\\ &= 1-\langle\varphi_{m,n}-P_S(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha} - \langle P_S(\varphi_{m,n})|\varphi_{m,n}-P_\Sigma(\varphi_{m,n})\rangle_{\nu_\alpha}.\end{aligned}$$

The relations (3.9) and (3.9) imply that

$$\begin{aligned}\langle\phi(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha} &= 1-\langle\varphi_{m,n}-P_S(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha} - \langle P_S(\varphi_{m,n})|\varphi_{m,n}-P_\Sigma(\varphi_{m,n})\rangle_{\nu_\alpha}\\ &\quad -\langle P_\Sigma P_S(\varphi_{m,n})|\varphi_{m,n}-P_S(\varphi_{m,n})\rangle_{\nu_\alpha}.\end{aligned}$$

Thus,

$$\begin{aligned}\langle\phi(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha} &= \Big|\langle\phi(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha}\Big|\\ &\geqslant 1-\Big|\langle\varphi_{m,n}-P_S(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha}\Big| - \Big|\langle P_S(\varphi_{m,n})|\varphi_{m,n}-P_\Sigma(\varphi_{m,n})\rangle_{\nu_\alpha}\Big|\\ &\quad -\Big|\langle P_\Sigma P_S(\varphi_{m,n})|\varphi_{m,n}-P_S(\varphi_{m,n})\rangle_{\nu_\alpha}\Big|.\end{aligned}\tag{3.9}$$

However, for every $(m,n)\in\mathbb{N}^2$,

$$\begin{aligned}a_{m,n}(S)=||\mathbf{1}_{S^c}\varphi_{m,n}||_{2,\nu_\alpha} &= ||\varphi_{m,n}-P_S\varphi_{m,n}||_{2,\nu_\alpha}\\ &\geqslant \Big|\langle\varphi_{m,n}-P_S\varphi_{m,n}|\varphi_{m,n}\rangle_{\nu_\alpha}\Big|,\end{aligned}\tag{3.10}$$

$$\begin{aligned}\Big|\langle P_\Sigma P_S\varphi_{m,n}|\varphi_{m,n}-P_S\varphi_{m,n}\rangle_{\nu_\alpha}\Big| &\leqslant ||P_\Sigma P_S||\;||\varphi_{m,n}-P_S\varphi_{m,n}||_{2,\nu_\alpha}\\ &\leqslant ||\varphi_{m,n}-P_S\varphi_{m,n}||_{2,\nu_\alpha}=a_{m,n}(S),\end{aligned}\tag{3.11}$$

$$\begin{aligned}\left|\langle P_S\varphi_{m,n}|\varphi_{m,n}-P_\Sigma\varphi_{m,n}\rangle_{\nu_\alpha}\right| &\leqslant ||P_S||\ ||\varphi_{m,n}-P_\Sigma\varphi_{m,n}||_{2,\nu_\alpha}\\ &\leqslant ||\mathscr{F}_\alpha(\varphi_{m,n}-P_\Sigma\varphi_{m,n})||_{2,\gamma_\alpha}=b_{m,n}(\Sigma). \end{aligned}\tag{3.12}$$

Combining the relations (3.9), (3.10), (3.11) and (3.12), we get

$$\langle\phi(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha}\geqslant 1-2a_{m,n}(S)-b_{m,n}(\Sigma). \tag{3.13}$$

Similarly, let R be the self adjoint operator defined by

$$R=(P_SP_\Sigma)^*(P_SP_\Sigma)=P_\Sigma P_SP_\Sigma,$$

by Theorem 3.3, we have

$$\begin{aligned} tr(R) &= \sum_{(m,n)\in\mathbb{N}^2}||P_SP_\Sigma(\varphi_{m,n})||^2_{2,\nu_\alpha}\\ &= \sum_{(m,n)\in\mathbb{N}^2}\langle R(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha}\\ &= ||P_SP_\Sigma||^2_{HS}\leqslant\nu_\alpha(S)\gamma_\alpha(\Sigma). \end{aligned}\tag{3.14}$$

As the same way, for every $(m,n)\in\mathbb{N}^2$, we have

$$\langle R(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha}\geqslant 1-a_{m,n}(S)-2b_{m,n}(\Sigma). \tag{3.15}$$

Using the relations (3.8), (3.13), (3.14) and (3.15), we deduce that

$$\begin{aligned} &\sum_{(m,n)\in\mathscr{K}}\left(2-3a_{m,n}(S)-3b_{m,n}(\Sigma)\right)\\ &\leqslant \sum_{(m,n)\in\mathscr{K}}\langle\phi(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha}+\sum_{(m,n)\in\mathscr{K}}\langle R(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha}\\ &\leqslant \sum_{(m,n)\in\mathbb{N}^2}\langle\phi(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha}+\sum_{(m,n)\in\mathbb{N}^2}\langle R(\varphi_{m,n})|\varphi_{m,n}\rangle_{\nu_\alpha}\\ &\leqslant 2\ \nu_\alpha(S)\ \gamma_\alpha(\Sigma). \end{aligned}\tag{3.16}$$

The proof of theorem is complete. □

COROLLARY 3.5. *Let ε, ρ and η be positive real numbers such that $0<\varepsilon<\frac{1}{3}$. Let $\mathscr{K}\subset\mathbb{N}^2$ be a nonempty subset and let $(\varphi_{m,n})_{(m,n)\in\mathscr{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. If for every $(m,n)\in\mathscr{K}$, $\varphi_{m,n}$ is $\varepsilon-$concentrated in the ball B^+_ρ and $\varphi_{m,n}$ is $\varepsilon-$bandlimited in the ball $\widetilde{B}^+_\eta$, then the subset $\mathscr{K}$ is finite and*

$$card(\mathscr{K})\leqslant\frac{\rho^{2\alpha+3}\ \eta^{2\alpha+3}}{(1-3\varepsilon)\ 2^{2\alpha+3}\left(\Gamma(\alpha+\frac{5}{2})\right)^2}. \tag{3.17}$$

Proof. Let $\mathscr{K}_1$ be a finite subset of $\mathscr{K}$. From the hypothesis, for every $(m,n)\in\mathscr{K}_1$,

$$a_{m,n}(B^+_\rho)=\left(\int\int_{(B^+_\rho)^c}\left|\varphi_{m,n}(r,x)\right|^2d\nu_\alpha(r,x)\right)^{\frac{1}{2}}\leqslant\varepsilon$$

and

$$b_{m,n}(\widetilde{B}_\eta^+) = \Big(\int\int_{(\widetilde{B}_\eta^+)^c} \big|\mathscr{F}_\alpha(\varphi_{m,n})(\mu,\lambda)\big|^2 d\gamma_\alpha(\mu,\lambda)\Big)^{\frac{1}{2}} \leqslant \varepsilon,$$

and consequently, for every $(m,n) \in \mathscr{K}_1$,

$$1 - \frac{3}{2}a_{m,n}(B_\rho^+) - \frac{3}{2}b_{m,n}(\widetilde{B}_\eta^+) \geqslant 1 - 3\varepsilon.$$

According to Theorem 3.4, it follows that

$$\begin{aligned}(1-3\varepsilon)\mathrm{card}(\mathscr{K}_1) &\leqslant \sum_{(m,n)\in \mathscr{K}_1} \Big(1 - \frac{3}{2}a_{m,n}(B_\rho^+) - \frac{3}{2}b_{m,n}(\widetilde{B}_\eta^+)\Big) \\ &\leqslant \nu_\alpha(B_\rho^+)\ \gamma_\alpha(\widetilde{B}_\eta^+).\end{aligned}$$

However, we have

$$\nu_\alpha(B_\rho^+) = \frac{\rho^{2\alpha+3}}{2^{\alpha+\frac{3}{2}}\ \Gamma(\alpha+\frac{5}{2})} \text{ and } \gamma_\alpha(\widetilde{B}_\eta^+) = \nu_\alpha\big(\theta(\widetilde{B}_\eta^+)\big) = \nu_\alpha(B_\eta^+) = \frac{\eta^{2\alpha+3}}{2^{\alpha+\frac{3}{2}}\ \Gamma(\alpha+\frac{5}{2})}.$$

This involves that for every finite subset $\mathscr{K}_1$ of $\mathscr{K}$, we have

$$\mathrm{card}(\mathscr{K}_1) \leqslant \frac{\rho^{2\alpha+3}\ \eta^{2\alpha+3}}{(1-3\varepsilon)\ 2^{2\alpha+3}\ \big(\Gamma(\alpha+\frac{5}{2})\big)^2}.$$

Consequently, $\mathscr{K}$ is a finite subset and

$$\mathrm{card}(\mathscr{K}) \leqslant \frac{\rho^{2\alpha+3}\ \eta^{2\alpha+3}}{(1-3\varepsilon)\ 2^{2\alpha+3}\ \big(\Gamma(\alpha+\frac{5}{2})\big)^2}. \quad \square$$

COROLLARY 3.6. *Let a,p be positive real numbers. Let $\mathscr{K}$ be a nonempty subset of $\mathbb{N}^2$ and let $(\varphi_{m,n})_{(m,n)\in \mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Assume that for every $(m,n) \in \mathscr{K}$,*

$$\rho_{p,\nu_\alpha}(\varphi_{m,n}) \leqslant a \text{ and } \rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big) \leqslant a.$$

Then $\mathscr{K}$ is a finite subset and

$$card(\mathscr{K}) \leqslant \frac{2^{\frac{8}{p}(2\alpha+3)-2\alpha-1} a^{2(2\alpha+3)}}{\big(\Gamma(\alpha+\frac{5}{2})\big)^2}.$$

Proof. Let ρ,η be positive real numbers, from the hypothesis, we deduce that for every $(m,n) \in \mathscr{K}$,

$$\int\int_{(B_\rho^+)^c} \big|\varphi_{m,n}(r,x)\big|^2 d\nu_\alpha(r,x) \leqslant \frac{1}{\rho^p}\int_0^\infty \int_{\mathbb{R}} \big|\varphi_{m,n}(r,x)\big|^2\ \big|(r,x)\big|^p\ d\nu_\alpha(r,x) \leqslant \Big(\frac{a}{\rho}\Big)^p$$

and

$$\begin{aligned}&\int\int_{(\widetilde{B}_\eta)^c}\left|\mathscr{F}_\alpha(\varphi_{m,n})(\lambda_0,\lambda)\right|^2 d\gamma_\alpha(\lambda_0,\lambda)\\&\leqslant \frac{1}{\eta^p}\int\int_{\Upsilon_+}|\theta(\lambda_0,\lambda)|^p\left|\mathscr{F}_\alpha(\varphi_{m,n})(\lambda_0,\lambda)\right|^2 d\gamma_\alpha(\lambda_0,\lambda)\\&\leqslant \left(\frac{a}{\eta}\right)^p.\end{aligned}$$

In particular, if we pick $\rho=\eta=a\,2^{\frac{4}{p}}$, we deduce that for every $(m,n)\in\mathscr{K}$; $\varphi_{m,n}$ is $\frac{1}{4}$-concentrated in the ball $B^+_{a2^{\frac{4}{p}}}$ and $\frac{1}{4}$-bandlimited in the ball $\widetilde{B}^+_{a2^{\frac{4}{p}}}$.

Applying Corollary 3.5, it follows that $\mathscr{K}$ is a finite subset of $\mathbb{N}^2$ and that

$$\operatorname{card}(\mathscr{K})\leqslant \frac{\left(a\,2^{\frac{4}{p}}\right)^{2\alpha+3}}{\left(1-\frac{3}{4}\right)2^{2\alpha+3}}\,\frac{\left(a\,2^{\frac{4}{p}}\right)^{2\alpha+3}}{\left(\Gamma(\alpha+\frac{5}{2})\right)^2}=\frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}\,a^{2(2\alpha+3)}}{\left(\Gamma(\alpha+\frac{5}{2})\right)^2}.\quad\square$$

LEMMA 3.7. *Let $p>0$ and let $(\varphi_{m,n})_{(m,n)\in\mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Then, there exists $j_0\in\mathbb{Z}$ such that*

$$\forall (m,n)\in\mathbb{N}^2,\ \max\left\{\rho_{p,\nu_\alpha}(\varphi_{m,n}),\ \rho_{p,\gamma_\alpha}\left(\mathscr{F}_\alpha(\varphi_{m,n})\right)\right\}\geqslant 2^{j_0}.$$

Proof. For every $j\in\mathbb{Z}$, let

$$P_j=\left\{(m,n)\in\mathbb{N}^2;\ 2^j\leqslant\max\left\{\rho_{p,\nu_\alpha}(\varphi_{m,n}),\ \rho_{p,\gamma_\alpha}\left(\mathscr{F}_\alpha(\varphi_{m,n})\right)\right\}<2^{j+1}\right\}.$$

Then, $\mathbb{N}^2=\bigcup_{j\in\mathbb{Z}}P_j$, $P_{j_1}\cap P_{j_2}=\emptyset$ if $j_1\neq j_2$ and for every $(m,n)\in P_j$,

$$\rho_{p,\nu_\alpha}(\varphi_{m,n})<2^{j+1}\text{ and }\rho_{p,\gamma_\alpha}\left(\mathscr{F}_\alpha(\varphi_{m,n})\right)<2^{j+1}.$$

Applying Corollary 3.6, we deduce that P_j is finite and

$$\operatorname{card}(P_j)\leqslant\frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\left(\Gamma(\alpha+\frac{5}{2})\right)^2}\left(2^{j+1}\right)^{4\alpha+6}.\tag{3.18}$$

Thus, for j negative and $|j|$ sufficiently large, we get $\operatorname{card}(P_j)=0$ or $P_j=\emptyset$. This means that there exists $j_0\in\mathbb{Z}$ such that $\forall j<j_0$, $P_j=\emptyset$. So,

$$\mathbb{N}^2=\bigcup_{j\in\mathbb{Z}}P_j=\bigcup_{j=j_0}^{+\infty}P_j.$$

This implies that

$$\forall (m,n)\in\mathbb{N}^2,\ \max\left\{\rho_{p,\nu_\alpha}(\varphi_{m,n}),\ \rho_{p,\gamma_\alpha}\left(\mathscr{F}_\alpha(\varphi_{m,n})\right)\right\}\geqslant 2^{j_0}.\quad\square$$

THEOREM 3.8. (Quantitative version of the mean-dispersion Shapiro's theorem) *Let* $(\varphi_{m,n})_{(m,n)\in\mathbb{N}^2}$ *be an orthonormal sequence in* $L^2(d\nu_\alpha)$*, then for every positive real number* p *and for every nonempty finite subset* $\mathscr{K}\subset\mathbb{N}^2$*, we have*

$$\sum_{(m,n)\in\mathscr{K}}\Big(\big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\big)^p+\big(\rho_{p,\gamma_\alpha}(\mathscr{F}_\alpha(\varphi_{m,n}))\big)^p\Big)$$
$$\geqslant\big(card(\mathscr{K})\big)^{1+\frac{p}{4\alpha+6}}\Big(\frac{\Gamma^2(\alpha+\frac{5}{2})(2^{4\alpha+6}-1)}{2^{(2\alpha+3)(\frac{10}{p}+3)+3}}\Big)^{\frac{p}{4\alpha+6}}. \tag{3.19}$$

Proof. Let j_0 be defined in Lemma 3.7. Then for every $(m,n)\in\mathbb{N}^2$,

$$\max\big\{\rho_{p,\nu_\alpha}(\varphi_{m,n}),\ \rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big\}\geqslant 2^{j_0}.$$

For every $k\geqslant j_0$, we put

$$Q_k=\bigcup_{j=j_0}^{k}P_j$$

From the relation (3.18),

$$\begin{aligned}\mathrm{card}(Q_k)&=\sum_{j=j_0}^{k}card(P_j)\\&\leqslant\frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\Big(\Gamma\Big(\alpha+\frac{5}{2}\Big)\Big)^2}\sum_{j=j_0}^{k}(2^{4\alpha+6})^{j+1}\\&=\frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\Big(\Gamma\Big(\alpha+\frac{5}{2}\Big)\Big)^2}\,2^{(4\alpha+6)(j_0+1)}\,\frac{2^{(4\alpha+6)(k-j_0+1)}-1}{2^{4\alpha+6}-1}\\&\leqslant\frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\Big(\Gamma\Big(\alpha+\frac{5}{2}\Big)\Big)^2(2^{4\alpha+6}-1)}\,2^{(4\alpha+6)(k+2)}.\end{aligned}\tag{3.20}$$

i) If $\mathrm{card}(\mathscr{K})>2\,\dfrac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\Big(\Gamma\Big(\alpha+\frac{5}{2}\Big)\Big)^2(2^{4\alpha+6}-1)}\,2^{(4\alpha+6)(j_0+2)}$; let $k>j_0$ such that

$$2\,\frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}2^{(4\alpha+6)(k+1)}}{\Big(\Gamma\Big(\alpha+\frac{5}{2}\Big)\Big)^2(2^{4\alpha+6}-1)}\leqslant\mathrm{card}(\mathscr{K})<2\,\frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}2^{(4\alpha+6)(k+2)}}{\Big(\Gamma\Big(\alpha+\frac{5}{2}\Big)\Big)^2(2^{4\alpha+6}-1)}. \tag{3.21}$$

From the relations (3.20) and (3.21, we have

$$\mathrm{card}(Q_{k-1})\leqslant\frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}2^{(4\alpha+6)(k+1)}}{\Big(\Gamma\Big(\alpha+\frac{5}{2}\Big)\Big)^2(2^{4\alpha+6}-1)}\leqslant\frac{\mathrm{card}(\mathscr{K})}{2}. \tag{3.22}$$

On the other hand,

$$\begin{aligned}
&\sum_{(m,n)\in \mathscr{K}} \Big(\big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\big)^p + \big(\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big)^p\Big)\\
&\qquad = \sum_{(m,n)\in \mathscr{K}\cap Q_{k-1}} \Big(\big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\big)^p + \big(\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big)^p\Big)\\
&\qquad\quad + \sum_{(m,n)\in \mathscr{K}\setminus Q_{k-1}} \Big(\big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\big)^p + \big(\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big)^p\Big)\\
&\qquad \geqslant \sum_{(m,n)\in \mathscr{K}\setminus Q_{k-1}} \Big(\big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\big)^p + \big(\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big)^p\Big).
\end{aligned}$$

But, for every $(m,n)\in \mathscr{K}\setminus Q_{k-1}$,

$$\begin{aligned}
\big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\big)^p + \big(\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big)^p &\geqslant \Big(\max\big\{\rho_{p,\nu_\alpha}(\varphi_{m,n}),\ \rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big\}\Big)^p.\\
&\geqslant 2^{kp}
\end{aligned}$$

So,

$$\sum_{(m,n)\in \mathscr{K}} \Big(\big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\big)^p + \big(\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big)^p\Big) \geqslant 2^{kp}\,\mathrm{card}(\mathscr{K}\setminus Q_{k-1}).$$

Then, from the relation (3.22), we deduce that

$$\sum_{(m,n)\in \mathscr{K}} \Big(\big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\big)^p + \big(\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big)^p\Big) \geqslant 2^{kp}\,\frac{\mathrm{card}(\mathscr{K})}{2}. \tag{3.23}$$

Now, from the relation (3.21), we have

$$\begin{aligned}
\frac{\mathrm{card}(\mathscr{K})\,2^{kp-1}}{\big(\mathrm{card}(\mathscr{K})\big)^{1+\frac{p}{4\alpha+6}}} &= 2^{kp-1}\big(\mathrm{card}(\mathscr{K})\big)^{-\frac{p}{4\alpha+6}}\\
&> 2^{kp-1}\left(2\,\frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\Big(\Gamma\Big(\alpha+\frac{5}{2}\Big)\Big)^2\,(2^{4\alpha+6}-1)}\,2^{(4\alpha+6)(k+2)}\right)^{-\frac{p}{4\alpha+6}}\\
&= \left(\frac{2^{\frac{(1-kp)(4\alpha+6)}{p}+1+\frac{8}{p}(2\alpha+3)-(2\alpha+1)+(k+2)(4\alpha+6)}}{\Big(\Gamma\Big(\alpha+\frac{5}{2}\Big)\Big)^2\,(2^{4\alpha+6}-1)}\right)^{-\frac{p}{4\alpha+6}}\\
&= \left(\frac{\Big(\Gamma\Big(\alpha+\frac{5}{2}\Big)\Big)^2\,(2^{4\alpha+6}-1)}{2^{(2\alpha+3)(3+\frac{10}{p})+3}}\right)^{\frac{p}{4\alpha+6}}
\end{aligned}$$

which means that

$$2^{kp-1}\mathrm{card}(\mathscr{K}) > \big(\mathrm{card}(\mathscr{K})\big)^{1+\frac{p}{4\alpha+6}} \left(\frac{\left(\Gamma\left(\alpha+\frac{5}{2}\right)\right)^2(2^{4\alpha+6}-1)}{2^{(2\alpha+3)(3+\frac{10}{p})+3}}\right)^{\frac{p}{4\alpha+6}}. \quad (3.24)$$

Combining the relations (3.23) and (3.24), we get

$$\sum_{(m,n)\in\mathscr{K}} \Big(\big(\rho_{p,v_\alpha}(\varphi_{m,n})\big)^p + \big(\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big)^p\Big)$$
$$\geqslant \big(\mathrm{card}(\mathscr{K})\big)^{1+\frac{p}{4\alpha+6}} \left(\frac{\left(\Gamma\left(\alpha+\frac{5}{2}\right)\right)^2(2^{4\alpha+6}-1)}{2^{(2\alpha+3)(3+\frac{10}{p})+3}}\right)^{\frac{p}{4\alpha+6}}.$$

ii) If $\mathrm{card}(\mathscr{K}) \leqslant 2\,\dfrac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\left(\Gamma(\alpha+\frac{5}{2})\right)^2(2^{4\alpha+6}-1)}\,2^{(4\alpha+6)(j_0+2)}$. By Lemma 3.7, we have

$$\sum_{(m,n)\in\mathscr{K}} \Big(\big(\rho_{p,v_\alpha}(\varphi_{m,n})\big)^p + \big(\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big)^p\Big)$$
$$\geqslant \sum_{(m,n)\in\mathscr{K}} \Big(\max\big\{\rho_{p,v_\alpha}(\varphi_{m,n}),\ \rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big\}\Big)^p$$
$$\geqslant \mathrm{card}(\mathscr{K})\,2^{j_0p}.$$

As the same way,

$$\frac{\mathrm{card}(\mathscr{K})\,2^{j_0p}}{\big(\mathrm{card}(\mathscr{K})\big)^{1+\frac{p}{4\alpha+6}}} = 2^{j_0p}\big(\mathrm{card}(\mathscr{K})\big)^{-\frac{p}{4\alpha+6}}$$
$$\geqslant 2^{j_0p}\left(2\,\frac{2^{\frac{8}{p}(2\alpha+3)-(2\alpha+1)}}{\left(\Gamma\left(\alpha+\frac{5}{2}\right)\right)^2(2^{4\alpha+6}-1)}\,2^{(4\alpha+6)(j_0+2)}\right)^{-\frac{p}{4\alpha+6}}$$
$$= \left(\frac{2^{-j_0(4\alpha+6)+1+\frac{8}{p}(2\alpha+3)-(2\alpha+3)+2+(j_0+2)(4\alpha+6)}}{\left(\Gamma\left(\alpha+\frac{5}{2}\right)\right)^2(2^{4\alpha+6}-1)}\right)^{-\frac{p}{4\alpha+6}}$$
$$= \left(\frac{2^{(2\alpha+3)(3+\frac{8}{p})+3}}{\left(\Gamma\left(\alpha+\frac{5}{2}\right)\right)^2(2^{4\alpha+6}-1)}\right)^{-\frac{p}{4\alpha+6}}$$
$$\geqslant \left(\frac{\left(\Gamma\left(\alpha+\frac{5}{2}\right)\right)^2(2^{4\alpha+6}-1)}{2^{(2\alpha+3)(3+\frac{10}{p})+3}}\right)^{\frac{p}{4\alpha+6}}.$$

Then,

$$\sum_{(m,n)\in\mathscr{K}}\Big(\big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\big)^p+\big(\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big)^p\Big)$$
$$\geqslant\big(\mathrm{card}(\mathscr{K})\big)^{1+\frac{p}{4\alpha+6}}\Big(\frac{\Gamma^2(\alpha+\frac{5}{2})(2^{4\alpha+6}-1)}{2^{(2\alpha+3)(\frac{10}{p}+3)+3}}\Big)^{\frac{p}{4\alpha+6}}.$$

REMARK 3.9.

i. The relation (3.19) shows in particular that for every orthonormal basis $(\varphi_{m,n})_{(m,n)\in\mathbb{N}^2}$ of $L^2(d\nu_\alpha)$ and for every $p>0$,

$$\sum_{(m,n)\in\mathbb{N}^2}\Big(\big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\big)^p+\big(\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\big)^p\Big)=+\infty.$$

ii. In [27, Remark 1], the authors have established the well known Heisenberg Pauli-Weyl uncertainty principle for the Riemann-Liouville transform, that is for every $f\in L^2(d\nu_\alpha)$, we have

$$||\,|(r,x)|\,f||^2_{2,\nu_\alpha}+||\,|\theta(\lambda_0,\lambda)|\,\mathscr{F}_\alpha(f)||^2_{2,\gamma_\alpha}\geqslant(2\alpha+3)\,||f||^2_{2,\nu_\alpha}. \quad (3.25)$$

Let $f\in L^2(d\nu_\alpha)\setminus\{0\}$ and let $\varphi_{0,0}=\dfrac{f}{||f||_{2,\nu_\alpha}}$. The set $\{\varphi_{0,0}\}$ can be completed to an Hilbert basis $(\varphi_{m,n})_{m,n\in\mathbb{N}^2}$ of $L^2(d\nu_\alpha)$. Taking $\mathscr{K}=\{(0,0)\}$ in the relation (3.19), we deduce that for every $p>0$,

$$||\,|(r,x)|^{\frac{p}{2}}\,f||^2_{2,\nu_\alpha}+||\,|\theta(\lambda_0,\lambda)|^{\frac{p}{2}}\,\mathscr{F}_\alpha(f)||^2_{2,\gamma_\alpha}$$
$$\geqslant\Big(\frac{\Gamma^2(\alpha+\frac{5}{2})(2^{4\alpha+6}-1)}{2^{(2\alpha+3)(\frac{10}{p}+3)+3}}\Big)^{\frac{p}{4\alpha+6}}\,||f||^2_{2,\nu_\alpha}. \quad (3.26)$$

The relation (3.26) generalizes the relation (3.25). However, in the relation (3.25), the constant $2\alpha+3$ is optimal (the best). □

LEMMA 3.10. *For every positive real number a, there exists a non zero function $f\in L^2(d\nu_\alpha)$ which vanishes almost everywhere on $B_a^+=\big\{(r,x)\in[0,+\infty[\times\mathbb{R};\ r^2+x^2\leqslant a^2\big\}$ and such that $\mathscr{F}_\alpha(f)$ vanishes almost every where on $\widetilde{B}_a^+=\big\{(\lambda_0,\lambda)\in\Upsilon_+;\ |\theta(\lambda_0,\lambda)|^2=\lambda_0^2+2\lambda^2\leqslant a^2\big\}$.*

Proof. Let $\mathscr{H}_\alpha$ be the Hankel transform defined on $L^1(d\mu_\alpha)\cap L^2(d\mu_\alpha)$ by

$$\mathscr{H}_\alpha(f)(\lambda_0)=\int_0^\infty f(r)j_\alpha(r\lambda_0)\,d\mu_\alpha(r)$$

where $d\mu_\alpha$ is the measure defined on $[0,+\infty[$ by $d\mu_\alpha(r)=\dfrac{r^{2\alpha+1}\,dr}{2^\alpha\Gamma(\alpha+1)}$. Let $a>0$, according to [12], there exists a nonzero function $g\in L^2(d\mu_\alpha)$ such that g and $\mathscr{H}_\alpha(g)$

vanish almost everywhere on $[0,a]$. As the same way and using [24], there exists a nonzero function $h \in L^2(\mathbb{R},dx)$ such that h and $\widehat{h}$ vanish almost everywhere on $[-a,a]$ where

$$\widehat{h}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(x)\, e^{-i\lambda x} dx.$$

We consider the function $f(r,x) = g(r)h(x)$. By construction, f belongs to $L^2(d\nu_\alpha)$ and f vanishes on B_a^+. Moreover, for every $(\lambda_0,\lambda) \in \Upsilon$,

$$\mathscr{F}_\alpha(f)(\lambda_0,\lambda) = \mathscr{H}_\alpha(g)\big(\sqrt{\lambda_0^2+\lambda^2}\big)\widehat{h}(\lambda),$$

consequently, $\mathscr{F}_\alpha(f)$ vanishes almost everywhere on

$$\widetilde{B}_a^+ = \{(\lambda_0,\lambda) \in \Upsilon_+;\ |\theta(\lambda_0,\lambda)|^2 = \lambda_0^2 + 2\lambda^2 \leqslant a^2\}. \quad \square$$

THEOREM 3.11. (Multiplicative version of the mean-dispersion Shapiro's theorem) *Let* $(\varphi_{m,n})_{(m,n)\in\mathbb{N}^2}$ *be an orthonormal basis of* $L^2(d\nu_\alpha)$. *For every* $p>0$, *the sequence* $\Big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\Big)_{(m,n)\in\mathbb{N}^2}$ *is not bounded, that is*

$$\sup_{(m,n)\in\mathbb{N}^2} \Big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\Big) = +\infty.$$

Proof. Suppose that

$$\sup_{(m,n)\in\mathbb{N}^2} \Big(\rho_{p,\nu_\alpha}(\varphi_{m,n})\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big)\Big) < +\infty.$$

Then, there exists a positive constant C such that

$$\forall (m,n) \in \mathbb{N}^2,\ \rho_{p,\nu_\alpha}(\varphi_{m,n})\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big) \leqslant C^2.$$

For every $k \in \mathbb{Z}$, we put

$$A_k = \Big\{(m,n) \in \mathbb{N}^2;\ 2^{-k}\, C \leqslant \rho_{p,\nu_\alpha}(\varphi_{m,n}) < 2^{-k+1}\, C\Big\},$$

then $A_{k_1} \cap A_{K_2} = \emptyset$ if $k_1 \neq k_2$ and $\bigcup_{k\in\mathbb{Z}} A_k = \mathbb{N}^2$.

Moreover, for every $(m,n) \in A_k$,

$$\rho_{p,\nu_\alpha}(\varphi_{m,n}) \leqslant 2^{-k+1}\, C \text{ and } \rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\varphi_{m,n})\big) \leqslant 2^k\, C. \tag{3.27}$$

On the other hand, for all $\rho,\eta > 0$ and $(m,n) \in A_k$, we have

$$\begin{aligned} \int\int_{(B_\rho^+)^c} \big|\varphi_{m,n}(r,x)\big|^2 d\nu_\alpha(r,x) &\leqslant \frac{1}{\rho^p} \int_0^\infty \int_{\mathbb{R}} \big|(r,x)\big|^p \big|\varphi_{m,n}(r,x)\big|^2 d\nu_\alpha(r,x) \\ &= \Big(\frac{\rho_{p,\nu_\alpha}(\varphi_{m,n})}{\rho}\Big)^p \leqslant \Big(\frac{2^{-k+1}\, C}{\rho}\Big)^p \end{aligned} \tag{3.28}$$

and

$$\int\int_{(\widetilde{B}_\eta^+)^c} \left|\mathscr{F}_\alpha(\varphi_{m,n})(\mu,\lambda)\right|^2 d\gamma_\alpha(\mu,\lambda) \leqslant \frac{1}{\eta^p}\left(\rho_{p,\gamma_\alpha}\left(\mathscr{F}_\alpha(\varphi_{m,n})\right)\right)^p$$
$$\leqslant \left(\frac{2^k C}{\eta}\right)^p. \qquad (3.29)$$

In particular, for $\rho = C\,2^{\frac{4}{p}+1}\,2^{-k}$ and $\eta = C\,2^{\frac{4}{p}}\,2^k$, we deduce that for every $(m,n) \in A_k$, the function $\varphi_{m,n}$ is $\frac{1}{4}$-concentrated in the ball $B^+_{C\,2^{\frac{4}{p}+1}2^{-k}}$ and $\frac{1}{4}$-bandlimited in the ball $\widetilde{B}^+_{C\,2^{\frac{4}{p}}\,2^k}$. Using Corollary 3.5, we conclude that the set A_k is finite and that

$$\begin{aligned}\mathrm{card}(A_k) &\leqslant \frac{\left(C\,2^{\frac{4}{p}+1}2^{-k}\right)^{2\alpha+3}\left(C\,2^{\frac{4}{p}}2^k\right)^{2\alpha+3}}{2^{2\alpha+3}\left(\Gamma\left(\alpha+\frac{5}{2}\right)\right)^2(1-\frac{3}{4})} \\ &= \frac{C^{2(2\alpha+3)}\,2^{\frac{8}{p}(2\alpha+3)+2}}{\left(\Gamma\left(\alpha+\frac{5}{2}\right)\right)^2}.\end{aligned} \qquad (3.30)$$

Now, let $R>0$. By Lemma 3.10, there exists $f \in L^2(d\nu_\alpha)$; $||f||_{2,\nu_\alpha} = 1$ such that f vanishes on B_R^+ and $\mathscr{F}_\alpha(f)$ vanishes on $\widetilde{B}_R^+$. For every $(m,n) \in A_k$, we have

$$\begin{aligned}\left|\langle f|\varphi_{m,n}\rangle_{\nu_\alpha}\right|^2 &\leqslant \left(\int\int_{(B_R^+)^c} \frac{|f(r,x)|}{|(r,x)|^{\frac{p}{2}}}\,|(r,x)|^{\frac{p}{2}}\,|\varphi_{m,n}(r,x)|d\nu_\alpha(r,x)\right)^2 \\ &\leqslant \left(\int\int_{(B_R^+)^c} \frac{|f(r,x)|^2}{|(r,x)|^p} d\nu_\alpha(r,x)\right)\left(\int_0^\infty\int_{\mathbb{R}} |(r,x)|^p\,|\varphi_{m,n}(r,x)|^2 d\nu_\alpha(r,x)\right) \\ &\leqslant \frac{1}{R^p}\,\rho^p_{p,\nu_\alpha}(\varphi_{m,n}).\end{aligned}$$

Using the relation (3.27), we get

$$\left|\langle f|\varphi_{m,n}\rangle_{\nu_\alpha}\right|^2 \leqslant \frac{(2C)^p\,2^{-kp}}{R^p}. \qquad (3.31)$$

Similarly, for every $(m,n) \in A_k$,

$$\begin{aligned}\left|\langle \mathscr{F}_\alpha(f)|\mathscr{F}_\alpha(\varphi_{m,n})\rangle_{\gamma_\alpha}\right|^2 &\leqslant \left(\int\int_{(\widetilde{B}_R^+)^c} \frac{|\mathscr{F}_\alpha(f)(\lambda_0,\lambda)|^2}{|\theta(\lambda_0,\lambda)|^p} d\gamma_\alpha(\lambda_0,\lambda)\right) \\ &\quad\times\left(\int\int_{\Upsilon_+} |\theta(\lambda_0,\lambda)|^p\,|\mathscr{F}_\alpha(\varphi_{m,n})(\lambda_0,\lambda)|^2 d\gamma_\alpha(\lambda_0,\lambda)\right) \\ &\leqslant \frac{1}{R^p}\,\rho^p_{p,\gamma_\alpha}\left(\mathscr{F}_\alpha(\varphi_{m,n})\right).\end{aligned}$$

Again, by the relation (3.27), we have

$$\left|\langle \mathscr{F}_\alpha(f)|\mathscr{F}_\alpha(\varphi_{m,n})\rangle_{\gamma_\alpha}\right|^2 \leqslant \frac{(2C)^p}{R^p}\,2^{kp}. \qquad (3.32)$$

Using the relations (3.31), (3.32) and the Parseval equality for $\mathscr{F}_\alpha$, it follows that for every $(m,n) \in A_k$,

$$\left|\langle f|\varphi_{m,n}\rangle_{\nu_\alpha}\right|^2 = \left|\langle \mathscr{F}_\alpha(f)|\mathscr{F}_\alpha(\varphi_{m,n})\rangle_{\gamma_\alpha}\right|^2 \leqslant \left(\frac{2C}{R}\right)^p \min\left\{2^{kp}, 2^{-kp}\right\} = \left(\frac{2C}{R}\right)^p 2^{-|k|\,p}. \tag{3.33}$$

On the other hand, we know that

$$\begin{aligned} ||f||^2_{2,\nu_\alpha} = 1 &= \sum_{(m,n)\in\,\mathbb{N}^2} \left|\langle f|\varphi_{m,n}\rangle_{\nu_\alpha}\right|^2 \\ &= \sum_{k\in\,\mathbb{Z}} \Big(\sum_{(m,n)\in\, A_k} \left|\langle f|\varphi_{m,n}\rangle_{\nu_\alpha}\right|^2 \Big). \end{aligned} \tag{3.34}$$

By the relations (3.30) and (3.33), we have

$$\begin{aligned} 1 &\leqslant \left(\frac{2C}{R}\right)^p \sum_{k\in\,\mathbb{Z}} 2^{-|k|p}\ \mathrm{card}(A_k) \\ &\leqslant \frac{C^{2(2\alpha+3)+p}\ 2^{\frac{8}{p}(2\alpha+3)+p+2}}{R^p\big(\Gamma(\alpha+\frac{5}{2})\big)^2}\Big(2\sum_{k=0}^{\infty}2^{-kp}-1\Big) \\ &= \frac{C^{2(2\alpha+3)+p}\ 2^{\frac{8}{p}(2\alpha+3)+p+2}}{R^p\big(\Gamma(\alpha+\frac{5}{2})\big)^2}\Big(\frac{2^p+1}{2^p-1}\Big). \end{aligned} \tag{3.35}$$

This gives a contradiction because we can choose R sufficiently large. $\square$

PROPOSITION 3.12. *There exists an orthonormal sequence* $(\psi_{m,n})_{(m,n)\in\,\mathbb{N}^2}$ *such that for every* $p>0$,

$$\sup_{(m,n)\in\,\mathbb{N}^2}\Big(\rho_{p,\nu_\alpha}(\psi_{m,n})\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\psi_{m,n})\big)\Big) < +\infty.$$

Proof. Let $\psi \in \mathscr{D}_e(\mathbb{R}^2)$; $\mathrm{supp}(\psi) \subset \{(r,x)\in\ \mathbb{R}^2;\ 1\leqslant |(r,x)|\leqslant 2\}$ such that $||\psi||_{2,\nu_\alpha}=1$. Since the transform $\widetilde{\mathscr{F}_\alpha}$ is a topological isomorphism from $\mathscr{S}_e(\mathbb{R}^2)$ onto itself, we deduce that $\widetilde{\mathscr{F}_\alpha}(\psi)$ belongs to $\mathscr{S}_e(\mathbb{R}^2)$. Consequently, from Definition 3.1 and the relation (2.19), for every $p>0$,

$$\rho_{p,\nu_\alpha}(\psi) < +\infty \text{ and } \rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\psi)\big) = \rho_{p,\nu_\alpha}\big(\widetilde{\mathscr{F}_\alpha}(\psi)\big) < +\infty\,.$$

Let $\theta:\ \mathbb{N}^2 \longrightarrow \mathbb{N}$ be a bijective application, we define the sequence $(\psi_{m,n})_{(m,n)\in\,\mathbb{N}^2}$ by

$$\forall (r,x)\in\ [0,+\infty[\times\mathbb{R},\ \psi_{m,n}(r,x) = 2^{(\alpha+\frac{3}{2})\theta(m,n)}\ \psi\big(2^{\theta(m,n)}r, 2^{\theta(m,n)}x\big).$$

For every $(m,n)\in\ \mathbb{N}^2$;

$$||\psi_{m,n}||_{2,\nu_\alpha} = 1,$$

and

$$\operatorname{supp}(\psi_{m,n}) \subset \left\{(r,x) \in \mathbb{R}^2;\ 2^{-\theta(m,n)} \leqslant |(r,x)| \leqslant 2^{1-\theta(m,n)}\right\}. \tag{3.36}$$

Let $(m,n),(m',n') \in \mathbb{N}^2$; $(m,n) \neq (m',n')$, then $\theta(m,n) \neq \theta(m',n')$, for example

$$\theta(m,n) < \theta(m',n'),\ \text{or } 1+\theta(m,n) \leqslant \theta(m',n').$$

Then, from the relation (3.36),

$$\forall (r,x) \in\ [0,+\infty[\times\mathbb{R};\ \psi_{m,n}(r,x)\psi_{m',n'}(r,x) = 0,$$

In particular $\langle\ \psi_{m,n}|\psi_{m',n'}\ \rangle_{\nu_\alpha} = 0$. Consequently, $(\psi_{m,n})_{(m,n)\in\ \mathbb{N}^2}$ is an orthonormal sequence in $L^2(d\nu_\alpha)$. On the other hand, by a standard computation, we have

$$\rho_{p,\nu_\alpha}(\psi_{m,n}) = 2^{-\theta(m,n)}\ \rho_{p,\nu_\alpha}(\psi) \tag{3.37}$$
$$\mathscr{F}_\alpha(\psi_{m,n})(\lambda_0,\lambda) = 2^{-(\alpha+\frac{3}{2})\theta(m,n)}\ \mathscr{F}_\alpha(\psi)\big(2^{-\theta(m,n)}\lambda_0, 2^{-\theta(m,n)}\lambda\big)$$

$$\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\psi_{m,n})\big) = 2^{\theta(m,n)}\ \rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\psi)\big). \tag{3.38}$$

The relations (3.37) and (3.38) show that

$$\forall (m,n) \in\ \mathbb{N}^2,\ \rho_{p,\nu_\alpha}(\psi_{m,n})\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\psi_{m,n})\big) = \rho_{p,\nu_\alpha}(\psi)\rho_{p,\gamma_\alpha}\big(\mathscr{F}_\alpha(\psi)\big). \quad \square$$

REFERENCES

[1] B. AMRI AND L. T. RACHDI, *The Littlewood-Paley g-function associated with the Riemann-Liouville operator*, Ann. Univ. Paedagog. Crac. Stud. Math. **12** (2013), 31–58.
[2] B. AMRI AND L. T. RACHDI, *Uncertainty principle in terms of entropy for the Riemann-Liouville operator*, Bull. Malays. Math. Sci. Soc., doi.org/101007/s-40840-015-0121-5.
[3] B. AMRI AND L. T. RACHDI, *Beckner Logarithmic Uncertainty principle for the Riemann-Liouville operator*, Internat. J. Math. **24**, no. 9 (2013) 1350070 (29 pages).
[4] B. AMRI AND L. T. RACHDI, *Calderon-reproducing formula for singular partial differential operators*, Integral Transforms Spec. Funct. **25**, no. 8 (2014) 597–611.
[5] C. BACCAR, N. B. HAMADI AND L. T. RACHDI, *Inversion formulas for the Riemann-Liouville transform and its dual associated with singular partial differential operators*, Int. J. Math. Math. Sci. **2006** (2006) 1–26.
[6] C. BACCAR, N. B. HAMADI AND L. T. RACHDI, *Best approximation for Weierstrass transform connected with Riemann-Liouville operator*, Commun. Math. Anal. **5**, no. 1, (2008) 65–83.
[7] C. BACCAR AND L. T. RACHDI, *Spaces of D_{L^p}-type and a convolution product associated with the Riemann-Liouville operator*, Bull. Math. Anal. Appl., vol. **1**, Iss. 3 (2009) 16–41.
[8] A. BEURLING, *The collected works of Arne Beurling*, Birkhäuser., vol. **1–2**, Boston, 1989.
[9] W. R. BLOOM AND H. HEYER, *Harmonic analysis of probability measures on hypergroups*, de Gruyter studies in mathematics 20, walter de Gruyter, Berlin-New York, 1995.
[10] A. BONAMI, B. DEMANGE, AND P. JAMING, *Hermite functions and uncertainty priciples for the Fourier and the widowed Fourier transforms*, Rev. Mat. Iberoamericana, **19** (2003) 23–55.
[11] M. G. COWLING AND J. F. PRICE, *Generalizations of Heisenberg's inequality in Harmonic analysis*, (Cortona, 1982), Lecture Notes in Math., **992** (1983) 443–449.
[12] L. DE BRANGES, *Self-reciprocal functions*, J. Math. Anal. Appl. **9** (1964) 433–457.

[13] A. ERDÉLYI et al., *Tables of integral transforms*, Mc Graw-Hill Book Compagny., vol. **2**, New York, 1954.

[14] A. ERDÉLYI, *Asymptotic expansions*, Dover publications, New-York, 1956.

[15] J. A. FAWCETT, *Inversion of n-dimensional spherical averages*, SIAM Journal on Applied Mathematics, no. **02**, (1985), 336–341.

[16] G. B. FOLLAND AND A. SITARAM, *The uncertainty principle: a mathematical survey*, J. Fourier Anal. Appl. **3** (1997) 207–238.

[17] G. H. HARDY, *A theorem concerning Fourier transform*, J. London. Math. Soc. **8** (1933) 227–231.

[18] S. HELGASON, *The Radon Transform*, Birkhäuser, 2nd edition, 1999.

[19] H. HELLSTEN AND L.-E. ANDERSSON, *An inverse method for the processing of synthetic aperture radar data*, Inverse Problems **3**, no. 1 (1987), 111–124.

[20] M. HERBERTHSON, *A numerical implementation of an inverse formula for CARABAS raw data*, Internal Report D30430-3.2, National Defense Research Institude, FOA, Box 1165; S-581 11, Linköping, 1986.

[21] KH. HLEILI, S. OMRI AND L. T. RACHDI, *Uncertainty principle for the Riemann-Liouville operator*, Cubo, vol. **13**, no. 03, (2011) 91–115.

[22] P. JAMING, A. M. POWELL, *Uncertainty principles for orthonormal sequences*, J. Funct. Anal. **243** (2007), 611–630.

[23] N. N. LEBEDEV, *Special Functions and Their Applications, Dover publications*, New-York, 1972.

[24] E. MALINNIKOVA, *Orthonormal sequences in* $L^2(\mathbb{R}^d)$ *and time frequency localization*, J. Fourier Anal. Appl. **16**, (2010), 983–1006.

[25] G. W. MORGAN, *A note on Fourier transforms*, J. London. Math. Soc. **9** (1934) 178–192.

[26] S. OMRI AND L. T. RACHDI, *An* $L^p - L^q$ *version of Morgan's theorem associated with Riemann-Liouville transform*, Int. J. Math. Anal., vol. **1**, no. 17 (2007) 805–824.

[27] S. OMRI AND L. T. RACHDI, *Heisenberg-Pauli-Weyl uncertainty principle for the Riemann-Liouville Operator*, J. Inequal. Pure and Appl. Math. **9** (2008), Iss. 3, Art 88.

[28] L. T. RACHDI AND A. ROUZ, *On the range of the Fourier transform connected with Riemann-Liouville operator*, Ann. Math. Blaise Pascal, vol. **16**, no. 2 (2009) 355–397.

[29] L. T. RACHDI, B. AMRI AND C. CHETTAOUI, L^p*-Boundedness for the Littlewood-Paley g-Function Connected with the Riemann-Liouville Operator*, KYUNGPOOK Math. J. **56** (2016), 185–220.

[30] H. S. SHAPIRO, *Uncertainty principles for bases in* $L^2(\mathbb{R})$, in: Proceedings of the Conference on Harmonic Analysis and Number Theory, CIRM, Marseille-Luminy, October 16–21 (2005).

[31] E. M. STEIN, *Interpolation of linear operator*, Trans. Amer. Math. Soc. **83**, no. 2 (1956) 482–492.

[32] E. M. STEIN AND G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University, New Jersey, 1971.

[33] K. TRIMÈCHE, *Transformation intégrale de Weyl et théorème de Paley-Wiener associés à un opérateur différentiel singulier sur* $(0,+\infty)$, J. Math. Pures Appl. **60** (1981) 51–98.

[34] K. TRIMÈCHE, *Inversion of the Lions translation operator using genaralized wavelets*, Appl. Comput. Harmon. Anal. **4** (1997) 97–112.

[35] G. N. WATSON, *A treatise on the theory of Bessel functions*, Cambridge univ. Press., 2nd ed, Cambridge 1959.

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