

## SELF-COMMUTATOR NORM OF HYPONORMAL TOEPLITZ OPERATORS

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*Abstract.* Chu and Khavinson recently obtained a lower bound for the norm of the self-commutator of a certain class of hyponormal Toeplitz operators on the Hardy space. Via a different approach, we offer a generalization of their result.

### 1. Introduction

We denote by  $\mathbb{D}$  the open unit disk in the complex plane and  $\partial\mathbb{D}$  its boundary, the unit circle. Recall that the Hardy space  $H^2$  is the closed subspace of  $L^2 = L^2(\partial\mathbb{D})$  consisting of all functions whose negative Fourier coefficients vanish. Let  $P : L^2 \rightarrow H^2$  denote the orthogonal projection. For a bounded function  $\varphi \in L^\infty = L^\infty(\partial\mathbb{D})$ , the Toeplitz operator  $T_\varphi : H^2 \rightarrow H^2$  is defined as

$$T_\varphi(u) = P(\varphi u) \text{ for all } u \in H^2.$$

The study of Toeplitz operators on the Hardy space was initiated by the seminal paper [3] of Brown and Halmos in the sixties.

Define a linear operator  $J : L^2 \rightarrow L^2$  by  $J(u)(z) = \bar{z}u(\bar{z})$  for  $u \in L^2$  and  $z \in \partial\mathbb{D}$ . It is immediate that  $J$  is a unitary operator on  $L^2$  and it is not hard to verify that  $J$  maps  $(H^2)^\perp$  onto  $H^2$ . For  $\varphi \in L^\infty$ , the Hankel operator  $H_\varphi : H^2 \rightarrow H^2$  is defined as

$$H_\varphi(u) = J(I - P)(\varphi u) \text{ for } u \in H^2.$$

We list below a few properties of Toeplitz and Hankel operators that shall be useful for us.

- (a) For any  $\varphi \in L^\infty$ , we have  $\|T_\varphi\| = \|\varphi\|_\infty$  and  $T_\varphi^* = T_{\bar{\varphi}}$ .
- (b) For  $f, g \in H^2$ ,

$$\langle T_\varphi f, g \rangle = \langle \varphi f, g \rangle.$$

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(c) If  $h$  belongs to  $H^\infty$  (which is  $H^2 \cap L^\infty$ ), then  $T_h = M_h$ , the operator of multiplication by  $h$ . In addition,

$$T_{\bar{h}}(1) = \overline{h(0)}.$$

(d) For any  $h_1, h_2 \in H^\infty$ ,

$$T_{\bar{h}_1 \varphi h_2} = T_{\bar{h}_1} T_\varphi T_{h_2}.$$

(e) For  $u, f \in H^\infty$ , we have  $H_f^*(u) = T_{\bar{z}} T_{u(\bar{z})}(f)$ .

A bounded linear operator  $T$  on a Hilbert space is *hyponormal* if its self-commutator  $[T^*, T] = T^*T - TT^*$  is positive. Recall that the spectrum of an operator  $T$ , denoted by  $\text{sp}(T)$ , is the set of all complex numbers  $\lambda$  for which  $T - \lambda I$  is not invertible, where  $I$  is the identity operator. The celebrated Putnam's Inequality [9] asserts that for any hyponormal operator  $T$ ,

$$\|[T^*, T]\| \leq \frac{\text{Area}(\text{sp}(T))}{\pi}.$$

For a function  $\varphi \in H^\infty$ , it is well known that the Toeplitz operator  $T_\varphi$  is hyponormal and  $\text{sp}(T_\varphi) = \overline{\varphi(\mathbb{D})}$ . A lower bound for the norm of the commutator  $[T_\varphi^*, T_\varphi]$  was obtained by D. Khavinson [7]. Combining with Putnam inequality, one deduces the isoperimetric inequality (see, e.g. [2]). Recently, several papers [1, 6, 8] have investigated similar problems for analytic Toeplitz operators on the Bergman space.

On the Hardy space, hyponormal Toeplitz operators were characterized by Cowen in [5].

**THEOREM 1. (Cowen)** *If  $\varphi$  is in  $L^\infty(\partial\mathbb{D})$ , where  $\varphi = f + \bar{g}$  for  $f, g$  in  $H^2$ , then  $T_\varphi$  is hyponormal if and only if*

$$g = c + T_{\bar{h}}f$$

for some constant  $c$  and some function  $h \in H^\infty$  with  $\|h\|_\infty \leq 1$ .

**REMARK 2.** In general, the function  $h$  is not unique. For more details on this, see the discussion following the proof of [5, Theorem 1].

Under the additional hypothesis  $h(0) = 0$ , Chu and Khavinson [4] recently obtained a lower bound for the norm of the self-commutator of  $T_\varphi$ .

**THEOREM 3. (Chu-Khavinson)** *If  $\varphi = f + \overline{T_{\bar{h}}f}$  for  $f, h$  in  $H^\infty$ ,  $\|h\|_\infty \leq 1$  and  $h(0) = 0$ , then*

$$\|[T_\varphi^*, T_\varphi]\| \geq \|f - f(0)\|_2^2 = \|P(\varphi) - \varphi(0)\|_2^2. \tag{1}$$

Due to the additional condition that  $h(0) = 0$ , Theorem 3 is not applicable in more general situations. As an example, take  $\varphi(z) = z + \bar{z}/2$ . A direct calculation shows that

$$[T_\varphi^*, T_\varphi] = \frac{3}{4}e_0 \otimes e_0,$$

where  $e_0(z) = 1$  for  $z \in \partial\mathbb{D}$ . Recall that for elements  $u, v$  in  $H^2$ , we use  $u \otimes v$  to denote the operator given by  $(u \otimes v)(x) = \langle x, v \rangle u$  for  $x \in H^2$ . We then have  $\|[T_\varphi^*, T_\varphi]\| = \frac{3}{4}$  while  $\|P(\varphi) - \varphi(0)\|_2 = 1$ . This shows that inequality (1) is false in this case. More generally, as we shall see later, Theorem 3 is not applicable if  $\varphi = f + \lambda \bar{f}$  with  $0 < |\lambda| < 1$  and  $f$  an inner function vanishing at the origin.

The purpose of this note is twofold. First, we offer an improved version of Theorem 3 which is applicable even in the case  $h(0) \neq 0$ . Second, we present a different and more operator-theoretic proof than that of Chu and Khavinson.

We state here our main result.

**THEOREM 4.** *Let  $\varphi = f + \overline{T_{\bar{h}}f}$  be a bounded harmonic function on the unit disk, where  $f, h \in H^\infty$  with  $\|h\|_\infty \leq 1$  and  $|h(0)| < 1$ . Put  $\psi = f - h(0)T_{\bar{h}}f$ . Then*

$$\|[T_\varphi^*, T_\varphi]\| \geq \frac{\|\psi - \psi(0)\|_2^2}{1 - |h(0)|^2}. \tag{2}$$

**REMARK 5.** In the case  $h(0) = 0$ , we see that (2) reduces to Chu-Khavinson’s result. In the case  $|h(0)| = 1$ , the function  $h$  is a unimodular constant function. It then follows that  $[T_\varphi^*, T_\varphi] = 0$ , which means that  $T_\varphi$  is normal. We assume  $|h(0)| < 1$  to avoid such trivial case.

**REMARK 6.** If we fix  $f$  and let  $h$  vary in the unit ball of  $H^\infty$ , the norm  $\|[T_\varphi^*, T_\varphi]\|$  is always bounded by  $\|f\|_\infty^2$  (which follows from (3) in Section 2). As a consequence, the right-hand side of (2) remains bounded. We provide here a more direct argument to explain this. From the definition of  $\psi$ , we have  $\psi = T_{1-h(0)\bar{h}}(f)$ . We then compute

$$\begin{aligned} \|\psi - \psi(0)\|_2^2 &\leq \|\psi\|_2^2 = \|T_{1-h(0)\bar{h}}f\|_2^2 = \|T_f(1 - h(0)\bar{h})\|_2^2 \\ &\leq \|f\|_\infty^2 \|1 - h(0)\bar{h}\|_2^2 = \|f\|_\infty^2 (1 + |h(0)|^2 \|h\|_2^2 - 2|h(0)|^2) \\ &\leq \|f\|_\infty^2 (1 - |h(0)|^2) \quad (\text{since } \|h\|_\infty \leq 1), \end{aligned}$$

which implies that the ratio  $\|\psi - \psi(0)\|_2^2 / (1 - |h(0)|^2)$  is bounded by  $\|f\|_\infty^2$ .

## 2. Proof of the main result

In this section we offer a proof of our result and discuss an application. We begin with a simple and probably well-known fact from Functional Analysis. We present here a quick proof.

**LEMMA 7.** *Let  $T$  be a positive operator on a Hilbert space  $\mathcal{H}$ . Then for any  $v \in \mathcal{H}$ , the operator  $S = \langle Tv, v \rangle T - (Tv) \otimes (Tv)$  is positive as well.*

*Proof.* Let  $T^{1/2}$  denotes the positive square root of  $T$ . For any  $u \in \mathcal{H}$ ,

$$\langle Su, u \rangle = \|T^{1/2}v\|^2 \|T^{1/2}u\|^2 - |\langle T^{1/2}v, T^{1/2}u \rangle|^2 \geq 0$$

by Cauchy-Schwarz's inequality. The conclusion of the lemma now follows.  $\square$

Lemma 7 provides us with the following immediate consequence for Toeplitz operators with holomorphic symbols.

**PROPOSITION 8.** *Suppose that  $h$  is a bounded holomorphic function with  $\|h\|_\infty \leq 1$  and  $|h(0)| < 1$ . Put  $\xi = (1 - \overline{h(0)}h)/\sqrt{1 - |h(0)|^2}$ . Then*

$$I - T_h T_{\bar{h}} \geq \xi \otimes \xi$$

on the Hardy space  $H^2$ .

*Proof.* Because  $\|h\|_\infty \leq 1$ , the operator  $T_{\bar{h}}$  has norm at most 1. This implies that  $I - T_h T_{\bar{h}} \geq 0$ . Applying Lemma 7 with  $T = I - T_h T_{\bar{h}}$  and  $\nu = 1$  gives

$$(1 - \|T_{\bar{h}}(1)\|^2)(I - T_h T_{\bar{h}}) - (1 - T_h T_{\bar{h}}(1)) \otimes (1 - T_h T_{\bar{h}}(1)) \geq 0.$$

Since  $T_{\bar{h}}1 = \overline{h(0)}$  and  $T_h T_{\bar{h}}1 = \overline{h(0)}h$ , we obtain the the desired operator inequality.  $\square$

We are now ready for the proof of our main result.

*Proof of Theorem 4.* Write  $\varphi = f + \bar{g}$  with  $g = T_{\bar{h}}f$ . We have  $H_{\bar{g}} = T_{\bar{h}^*}H_{\bar{f}}$  (see [5, p. 811]), where  $h^*(z) = \overline{h(\bar{z})}$ . We now compute

$$\begin{aligned} [T_\varphi^*, T_\varphi] &= [T_{\bar{f}+g}^*, T_{f+\bar{g}}] = [T_{\bar{f}}, T_f] - [T_{\bar{g}}, T_g] \\ &= H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}} = H_{\bar{f}}^* (I - T_{h^*} T_{\bar{h}^*}) H_{\bar{f}}. \end{aligned} \quad (3)$$

Proposition 8 then implies

$$[T_\varphi^*, T_\varphi] \geq H_{\bar{f}}^*(\eta \otimes \eta) H_{\bar{f}} = H_{\bar{f}}^*(\eta) \otimes H_{\bar{f}}^*(\eta),$$

where

$$\eta = \frac{1 - \overline{h^*(0)}h^*}{\sqrt{1 - |h^*(0)|^2}} = \frac{1 - h(0)h^*}{\sqrt{1 - |h(0)|^2}}.$$

As a consequence,

$$\|[T_\varphi^*, T_\varphi]\| \geq \|H_{\bar{f}}^*(\eta) \otimes H_{\bar{f}}^*(\eta)\| = \|H_{\bar{f}}^*(\eta)\|_2^2. \quad (4)$$

On the other hand,

$$\begin{aligned} H_{\bar{f}}^*(\eta) &= T_{\bar{z}} T_{\eta(\bar{z})}(f) = \frac{1}{\sqrt{1 - |h(0)|^2}} T_{\bar{z}}(f - h(0)T_{\bar{h}}f) \\ &= \frac{1}{\sqrt{1 - |h(0)|^2}} T_{\bar{z}}(f - h(0)g) \quad (\text{since } g = T_{\bar{h}}f) \\ &= \frac{1}{\sqrt{1 - |h(0)|^2}} T_{\bar{z}}\psi, \end{aligned}$$

since  $\psi = f - h(0)g$ . We then have

$$\|H_{\bar{f}}^*(\eta)\| = \frac{\|T_{\bar{z}}\psi\|_2^2}{1 - |h(0)|^2} = \frac{\|\psi - \psi(0)\|_2}{\sqrt{1 - |h(0)|^2}},$$

which, together with (4), gives the required inequality (2)

$$\|[T_{\varphi}^*, T_{\varphi}]\| \geq \frac{\|\psi - \psi(0)\|_2^2}{1 - |h(0)|^2}. \quad \square$$

Theorem 4 provides a lower bound for the norm of  $\|[T_{\varphi}^*, T_{\varphi}]\|$  in terms of both the holomorphic and anti-holomorphic parts of  $\varphi$ . In the following corollary, we obtain a weaker estimate which only depends on the holomorphic part of  $\varphi$ .

**COROLLARY 9.** *Let  $\varphi = f + \overline{T_{\bar{h}}f}$  be a bounded harmonic function on the unit disk, where  $f, h \in H^{\infty}$  with  $\|h\|_{\infty} \leq 1$  and  $|h(0)| < 1$ . Then*

$$\|[T_{\varphi}^*, T_{\varphi}]\| \geq \frac{(1 - |h(0)| \|h\|_{\infty})^2}{1 - |h(0)|^2} \|f - f(0)\|_2^2. \quad (5)$$

*Proof.* Let us estimate the numerator of the right hand side of (2) in Theorem 4.

$$\begin{aligned} \|\psi - \psi(0)\|_2 &= \|(f - f(0)) - h(0)(g - g(0))\|_2 \\ &\geq \|f - f(0)\|_2 - |h(0)| \cdot \|g - g(0)\|_2 \\ &= \|f - f(0)\|_2 - |h(0)| \cdot \|T_{\bar{z}}g\|_2 \\ &= \|f - f(0)\|_2 - |h(0)| \cdot \|T_{\bar{z}}T_{\bar{h}}(f)\|_2 \\ &= \|f - f(0)\|_2 - |h(0)| \cdot \|T_{\bar{z}}T_{\bar{h}}(f - f(0))\|_2 \quad (\text{since } T_{\bar{z}}(f(0)) = 0) \\ &\geq \|f - f(0)\|_2 - |h(0)| \|h\|_{\infty} \cdot \|f - f(0)\|_2 \quad (\text{since } \|T_{\bar{h}}T_{\bar{z}}\| \leq \|h\|_{\infty}). \end{aligned}$$

Combing with (2) gives

$$\frac{\|\psi - \psi(0)\|_2^2}{1 - |h(0)|^2} \geq \frac{(1 - |h(0)| \|h\|_{\infty})^2}{1 - |h(0)|^2} \|f - f(0)\|_2^2$$

as desired.  $\square$

**EXAMPLE 10.** Consider  $\varphi(z) = z + \bar{z}/2$  so that  $f(z) = z$  and  $h(z) = \frac{1}{2}$ . We have seen in Introduction that  $\|[T_{\varphi}^*, T_{\varphi}]\| = \frac{3}{4}$ . On the other hand, the right-hand side of (5) is also equal to  $\frac{3}{4}$ . Consequently, (5) is in fact an equality in this case.

Using Putnam’s Inequality and Corollary 9, we obtain

**COROLLARY 11.** *If  $\varphi = f + \overline{T_{\bar{h}}f}$  for  $f, h \in H^{\infty}$  with  $\|h\|_{\infty} \leq 1$  and  $|h(0)| < 1$ , then*

$$\text{Area}(\text{sp}(T_{\varphi})) \geq \pi \frac{(1 - |h(0)| \|h\|_{\infty})^2}{1 - |h(0)|^2} \|f - f(0)\|_2^2.$$

Given a bounded function  $\varphi$  for which  $T_\varphi$  is hyponormal, the existence of the function  $h$  in the representation  $\varphi = f + \overline{T_{\bar{h}}f}$  is not unique. Our lower estimate of the norm of  $[T_\varphi^*, T_\varphi]$  in Theorem 4 depends on the value of  $h(0)$ . In some cases, it turns out that  $h(0)$  is independent of the choice of  $h$ . We illustrate this in the following example.

EXAMPLE 12. Let  $\chi$  be an inner function with  $\chi(0) = 0$ . Suppose  $f$  is a non-constant polynomial of  $\chi$  and  $g$  belongs to  $H^\infty$  such that  $T_{f+\bar{g}}$  is hyponormal. Then for any function  $h \in H^\infty$  satisfying  $\|h\|_\infty \leq 1$  and  $g = c + T_{\bar{h}}f$  for some constant  $c$ , the value  $h(0)$  is independent of  $h$ . In the case  $f = \chi$ , we have  $h(0) = \langle f, g \rangle$ .

*Proof.* Since  $f$  is a non-constant polynomial of  $\chi$ , there exist  $M \geq 1$  and complex numbers  $c_1, \dots, c_M$  such that  $c_M \neq 0$  and

$$f = c_0 + \dots + c_M \chi^M.$$

We then compute

$$\begin{aligned} \langle \chi^M, g \rangle &= \langle \chi^M, T_{\bar{h}}f + c \rangle = \langle \chi^M, T_{\bar{h}}f \rangle \quad (\text{since } \chi^M(0) = 0) \\ &= \langle \chi^M h, f \rangle = \sum_{j=0}^M \bar{c}_j \langle \chi^M h, \chi^j \rangle = \bar{c}_M h(0) = \langle \chi^M, f \rangle h(0). \end{aligned}$$

It follows that

$$h(0) = \frac{\langle \chi^M, g \rangle}{\langle \chi^M, f \rangle},$$

which is independent of the choice of  $h$ . In the case  $f = \chi$ , we have  $M = 1$  and hence  $h(0) = \langle f, g \rangle$ .  $\square$

REMARK 13. The lower estimate in Theorem 4 makes use of the value of  $h$  at the origin. For any  $a \in \mathbb{D}$ , we briefly discuss here how an estimate involving  $h(a)$  may be obtained. However, the formula is a little more complicated. Recall that  $k_a(z) = \sqrt{1 - |a|^2} / (1 - \bar{a}z)$  is the normalized reproducing kernel of the Hardy space at  $a$ . We shall write

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

for the Mobius automorphism of the unit disk that interchanges  $a$  and the origin. Note that  $\varphi_a \circ \varphi_a(z) = z$  for all  $z \in \mathbb{D}$ . Define the operator  $W_a$  by

$$W_a(u) = k_a \cdot (u \circ \varphi_a), \quad u \in L^2(\partial\mathbb{D}).$$

A change-of-variables on the unit circle shows that that  $W_a$  is a unitary operator on  $L^2(\partial\mathbb{D})$ . It is well known that  $H^2$  is a reducing subspace of  $W_a$  and

$$W_a^* T_\varphi W_a = T_{\varphi \circ \varphi_a}$$

for any bounded  $\varphi$ . As a consequence,  $T_\varphi$  is hyponormal if and only if  $T_{\varphi \circ \varphi_a}$  is hyponormal and their self-commutators have the same norm. Note that if  $g = T_{\bar{h}}f$

then it can be checked that  $g \circ \varphi_a = T_{h \circ \varphi_a}^-(f \circ \varphi_a) + c$  for some constant  $c$ . Applying Theorem 2 for  $\varphi \circ \varphi_a$  gives

$$\begin{aligned} \|[T_{\varphi \circ \varphi_a}^*, T_{\varphi \circ \varphi_a}]\| &\geq \frac{\|f \circ \varphi_a - (h \circ \varphi_a(0))g \circ \varphi_a\|_2^2 - |f \circ \varphi_a(0) - (h \circ \varphi_a(0))g \circ \varphi_a(0)|^2}{1 - |h \circ \varphi_a(0)|^2} \\ &= \frac{\|f \circ \varphi_a - h(a)g \circ \varphi_a\|_2^2 - |f(a) - h(a)g(a)|^2}{1 - |h(a)|^2}. \end{aligned} \tag{6}$$

Since  $W_a$  is a unitary operator, the first term in the numerator of (6) is equal to

$$\|W_a(f \circ \varphi_a - h(a)g \circ \varphi_a)\|_2^2 = \|(f - h(a)g)k_a\|_2^2.$$

We then have

$$\|[T_\varphi^*, T_\varphi]\| = \|[T_{\varphi \circ \varphi_a}^*, T_{\varphi \circ \varphi_a}]\| \geq \frac{\|(f - h(a)g)k_a\|_2^2 - |f(a) - h(a)g(a)|^2}{1 - |h(a)|^2}. \tag{7}$$

It is possible to obtain estimate (7) by following the proof of Theorem 4. One needs to modify Proposition 8 by setting  $\nu = k_{\bar{a}}$  instead of  $\nu = 1$ . However, some parts of calculation are a bit more complicated. We leave this for the interested reader.

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