Operators and Matrices
Volume 13, Number 1 (2019), 93–110
doi:10.7153/oam-2019-13-06

COMPACTNESS OF OPERATOR INTEGRATORS

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(*Communicated by H. Radjavi*)

Abstract. A function f from a closed interval $[a,b]$ to a Banach space X is a *regulated function* if one-sided limits of f exist at every point. A function α from $[a,b]$ to the space $\mathfrak{B}(X,Y)$, of bounded linear transformations form X to a Banach space Y, is said to be an *integrator* if for each X-valued regulated function f, the Riemann-Stieltjes sums (with sampling points in the interior of subintervals) of f with respect to α converge in Y. We use elementary methods to establish criteria for an integrator α to induce a compact linear transformation from the space, $\mathrm{Reg}(X)$, of X-valued regulated functions to Y. We give direct and elementary proofs for each result to be used, including, among other things, the fact that each integrator α induces a bounded linear transformation, $\widehat{\alpha}$, from $\mathrm{Reg}(X)$ to Y, and other folklore or known results which required reading large amount of literature.

1. Introduction

A function f from a closed interval $[a,b]$ to a Banach space X, is said to be *regulated* if one-sided limits of f exist at every point of $[a,b]$ [3, §7.6, p. 139]. The space, $\mathrm{Reg}(X)$, of regulated functions has been extensively studied [10, 5, 9, 8, 7, 2, 6, 1, 4], mostly in the case of one-dimensional X. We focus here on the integrators of the regulated functions. Given a function α from $[a,b]$ to the space $\mathfrak{B}(X,Y)$, of bounded linear transformations from X to a Banach space Y, and an $f \in \mathrm{Reg}(X)$, a natural question is whether or when there is a vector in Y to which the internal (sampling points in the interior of subintervals) Riemann-Stieltjes sums of f with respect to α converge. A function α with this property for each $f \in \mathrm{Reg}(X)$ will be called an *integrator*. We use elementary methods to ultimately establish criteria for an integrator α to induce a compact linear transformation from $\mathrm{Reg}(X)$ to Y. We also use very elementary methods to prove all necessary steps leading from the definition all the way to criteria for compactness of integrator induced operators. In particular we show that each integrator induces a bounded linear transformation from the space of regulated functions to Y. We are grateful to the referee for pointing out that this is a consequence of [6, Ch I, Th 4.20]. But that theorem requires quite a lot of reading and digging into the book. Furthermore it also uses another theorem in another book of Hönig in the proof. Since our emphases are elementary methods and direct leads to the final results, we give a very direct and elementary proof of this result, making it accessible

Mathematics subject classification (2010): Primary 46G10, Secondary 28B05.
Keywords and phrases: Banach space, operator, regulated function, integrator, semivariation.
The firs author supported by Thailand Research Grant #TRG5880051.

Paper OaM-13-06

to anyone with basic knowledge of functional analysis. Furthermore the proof includes a nice application of the uniform boundedness principle. In the meantime this also makes the exposition self-contained.

The paper is organized as follows. We introduce the definition and an equivalent formulation for regulated functions in section 2. In section 3, we introduce the notion of integrator. We give an elementary proof, using the uniform boundedness principle, of the fact that each integrator induces a bounded linear operator. We also use elementary methods to complete the circle by establishing an equivalent formulation of integrators. These facts are buried in many pages in the text [6]. In section 4, we establish criteria for the integrator induced operators to be compact.

2. Regulated functions

Fix real numbers $a<b$ and a Banach space X. A function $f:[a,b]\to X$ is said to be *regulated* (see [3, §7.6, p. 139] and [6, p. 16]) if one-sided limits $f(c^{+}):=\lim_{t\to c^{+}} f(t)$ exist for all $c\in[a,b)$, and $f(c^{-}):=\lim_{t\to c^{-}} f(t)$ exist for all $c\in(a,b]$. Denote by $\mathrm{Reg}([a,b],X)=\mathrm{Reg}(X)$ the space of X-valued regulated functions on $[a,b]$.

A *partition* P of the interval $[a,b]$ is given by a finite number of *division points* in $[a,b]$:

$$P\left(a=t_0<t_1<t_2<\cdots<t_{n(P)}=b\right), \qquad n(P)\in\mathbb{N}.$$

The set of all partitions of $[a,b]$ is denoted by $\mathscr{P}[a,b]$ or simply $\mathscr{P}$, whenever no confusion can arise.

A function $g:[a,b]\to X$ is a *step function* if there are

$$\text{a partition } P\left(a=t_0<t_1<t_2<\cdots<t_{n(P)}=b\right) \text{ and vectors } x_j\in X,\ 1\leqslant j\leqslant n(P)$$

such that $g(t)=x_j$ for all $t\in(t_{j-1},\,t_j)$, $1\leqslant j\leqslant n(P)$; i.e., g takes on constant values on each open subinterval in the partition. Since P and $n(P)$ are determined by g, we also denote $n(P)$ by $n(g)$. Regulated functions have the following useful characterization. It is not hard to see that step functions and uniform limits of step functions are regulated. The converse is also true.

THEOREM 1. [3, Th 7.6.1, p. 139] *A function $f:[a,b]\to X$ is regulated iff there exists a sequence $\{h_n\}$ of step functions, from $[a,b]$ to X, such that*

$$\lim_{n\to\infty}\left[\sup_{t\in[a,b]}\|h_n(t)-f(t)\|\right]=0 \qquad (\textit{i. e., } h_n\to f \textit{ uniformly on } [a,b].)$$

Furthermore, if $f\in\mathrm{Reg}(X)$, then

$$\|f\|:=\sup_{t\in[a,b]}\|f(t)\|<\infty,$$

and $(\mathrm{Reg}(X),\|\cdot\|)$ is a Banach space.

3. Integrators as bounded linear transformations

We begin the section with a precise definition of an integrator. Then we use elementary methods, including uniform boundedness principle, to prove that each integrator induces a bounded linear transformation. We are grateful to the referee for pointing out that this is a consequence of [6, Ch I, Th 4.20]. Since that theorem requires quite a lot of reading and digging into the book, and furthermore, in the proof, it also uses another theorem in another book of Hönig. Here we give a more elementary and direct proof, not involving the complication of the generality in [6].

The close unit ball of radius $r > 0$, centered at 0, of a normed vector space Z will be denoted by $[Z]_r$. Given Banach spaces X and Y, let $\mathfrak{B}(X,Y)$ denote the space of bounded linear transformations from X to Y. A function $\alpha : [a,b] \to \mathfrak{B}(X,Y)$ is called an *integrator* for the regulated functions if for each $f \in \mathrm{Reg}(X)$, there is a $y \in Y$ that satisfies the following condition:

(†) for every $\varepsilon > 0$ there is a partition $P_\varepsilon \left(a = s_0 < s_1 < s_2 < \cdots < s_{n(P_\varepsilon)} = b\right)$ such that for every partition $P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$ that refines

$$P_\varepsilon : \{s_k : 1 \leqslant k \leqslant n(P_\varepsilon)\} \subseteq \{t_j : 1 \leqslant j \leqslant n(P)\}$$

and for all choices of $t_j^* \in (t_{j-1}, t_j)$, $1 \leqslant j \leqslant n(P)$,

$$\left\| y - \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) \right\| < \varepsilon.$$

Given an integrator α and an $f \in \mathrm{Reg}(X)$, a routine verification reveals that the vector $y \in Y$ associated with f by α is unique. The vector y is called the *interior integral* (or *Dushnik integral* [6, p. 7]) of f with respect to α and is denoted by

$$\int_a^b [d\alpha(t)](f(t)) = y.$$

We now show, directly (without using any other results, beyond the uniform boundedness principle), that each integrator induces a bounded linear transformation from $\mathrm{Reg}(X)$ to Y.

THEOREM 2. *Let* $\alpha : [a,b] \to \mathfrak{B}(X,Y)$ *be an integrator for* $\mathrm{Reg}(X)$. *Then the map* $\widehat{\alpha}$ *defined by*

$$\widehat{\alpha}(f) = \int_a^b [d\alpha(t)](f(t)), \qquad f \in \mathrm{Reg}(X)$$

is a bounded linear transformation from $\mathrm{Reg}(X)$ *to* Y.

Proof. We omit the routine verification of linearity of $\widehat{\alpha}$. First we show that α is a bounded function. Since $\alpha_0 = \alpha - \alpha(a)$ and α give the same integral for each function in $\mathrm{Reg}(X)$. We assume, with no loss, that $\alpha(a) = 0$. Suppose α is not bounded. Then, inductively, there is a sequence $\{t_k\}_{k\in\mathbb{N}}$ in $[a,b]$ such that

$$\|\alpha(t_1)\| > 4, \qquad \|\alpha(t_k)\| > 2^{2^k} \|\alpha(t_{k-1})\| \quad \text{for all } k \geqslant 2.$$

By compactness, we may assume without loss of generality, that the sequence $\{t_k\}$ is monotonically increasing to a limit $t_\infty \in [a,b]$. (The decreasing case is handled similarly.) For each $k \geqslant 1$ there is an $x_k \in [X]_1$ such that

$$\|[\alpha(t_1)]x_1\| > 4, \quad \|[\alpha(t_k)]x_k\| > \max\left\{2^{2^k} \|\alpha(t_{k-1})\|, \frac{3}{4}\|\alpha(t_k)\|\right\} \quad \forall\ k \geqslant 2.$$

Let $t_0 = a$. For each $k \in \mathbb{N}$, define f_k by putting $f_k(t) = 2^{-j} x_j$ for $t_{j-1} < t < t_j$, $1 \leqslant j \leqslant k$, and $f_k(t) = 0$ for all other $t \in [a,b]$. Then for $k < l$ in $\mathbb{N}$, we have

$$\|f_l - f_k\| = \max\left\{2^{-j} \|x_j\| : k+1 \leqslant j \leqslant l\right\} \leqslant 2^{-k-1} \to 0 \text{ as } k,\ l \to \infty.$$

Thus there is a function $f \in \mathrm{Reg}(X)$ such that $\|f_k - f\| \to 0$. Note that $f(t) = f_k(t)$ for all $t \in [a, t_k]$, since $f_k|_{[a,t_k]} = f_l|_{[a,t_k]}$ for all $l \geqslant k$. Furthermore, since $t_k \nearrow t_\infty$, and each $f_k|[t_\infty, b] = 0$, $f|[t_\infty, b] = 0$. We may assume without loss of generality that $t_\infty = b$.

Let $y \in Y$. (We show that for every partition P_0 there are a refinement P of P_0 and interior sampling points such that the associated Riemann-Stieltjes sum of f differs in norm more than 1 from y. Therefore y cannot be the interior integral of f, for any $y \in Y$, and hence the interior integral of f does not exist.) Let

$$P_0\left(a = s_0 < s_1 < s_2 < \cdots < s_{n(P_0)} = b\right)$$

be an arbitrarily given partition. Since $t_l \nearrow t_\infty$, which is assumed to be b, there is a $k \in \mathbb{N}$ such that $2^k > \|y\| + \|\alpha(b)\| + 1$ and $t_k > s_{n(P_0)-1} \geqslant t_{k-1}$. Let

$$P\left(a = u_0 < u_1 < u_2 < \cdots < u_{n(P)} = b\right)$$

be the partition that satisfies

$$\{s_j :\ 1 \leqslant j \leqslant n(P_0)\} \cup \{t_l :\ 1 \leqslant l \leqslant k\} = \{u_i :\ 1 \leqslant i \leqslant n(P)\}.$$

Then $u_{n(P)-1} = t_k$. Arbitrarily choose $u_i^* \in (u_{i-1}, u_i)$ for $1 \leqslant i < n(P) - 1$; and $u_{n(P)}^* \in (t_k, t_{k+1})$. Let $t_0 = a$. Break up the sum

$$\sum_{i=1}^{n(P)} [\alpha(u_i) - \alpha(u_{i-1})](f(u_i^*))$$

according to which (t_{j-1},t_j) the interval (u_{i-1},u_i) is contained in. Since each f_l is 0 on the interval $[a,t_1]=[t_0,t_1]$, and so is f, we have

$$\begin{aligned}
&\left\| y-\sum_{i=1}^{n(P)}[\alpha(u_i)-\alpha(u_{i-1})](f(u_i^*))\right\| \\
\geqslant &\left\| \sum_{j=2}^{k}\left[\sum_{(u_{i-1},u_i)\subseteq(t_{j-1},t_j)}[\alpha(u_i)-\alpha(u_{i-1})](f(u_i^*))\right]\right. \\
&\qquad\qquad \left. +[\alpha(b)-\alpha(t_k)](f(u_{n(P)}^*))\right\| -\|y\| \\
= &\left\| \sum_{j=2}^{k}\left[\sum_{(u_{i-1},u_i)\subseteq(t_{j-1},t_j)}[\alpha(u_i)-\alpha(u_{i-1})](2^{-j}x_j)\right]\right. \\
&\qquad\qquad \left. +[\alpha(b)-\alpha(t_k)](2^{-k-1}x_{k+1})\right\| -\|y\| \\
= &\left\| \sum_{j=2}^{k}\left[\sum_{(u_{i-1},u_i)\subseteq(t_{j-1},t_j)}[\alpha(u_i)-\alpha(u_{i-1})]\right](2^{-j}x_j)\right. \\
&\qquad\qquad \left. +[\alpha(b)-\alpha(t_k)](2^{-k-1}x_{k+1})\right\| -\|y\| \\
= &\left\| \sum_{j=2}^{k}[\alpha(t_j)-\alpha(t_{j-1})](2^{-j}x_j)+[\alpha(b)-\alpha(t_k)](2^{-k-1}x_{k+1})\right\| -\|y\| \\
\geqslant &\left\| [\alpha(t_k)](2^{-k}x_k)\right\| -\left\| [\alpha(t_{k-1})](2^{-k}x_k)\right\| \\
&\quad -\sum_{j=2}^{k-1}\left[\left\| [\alpha(t_j)](2^{-j}x_j)\right\| +\left\| [\alpha(t_{j-1})](2^{-j}x_j)\right\| \right] \\
&\qquad -\left\| [\alpha(b)](2^{-k-1}x_{k+1})\right\| -\left\| [\alpha(t_k)](2^{-k-1}x_{k+1})\right\| -\|y\| \\
> &\, 2^{-k}\left[\frac{3}{4}\|\alpha(t_k)\|\right]-2^{-k}\|\alpha(t_{k-1})\|-2^{-k-1}\|\alpha(b)\|-2^{-k-1}\|\alpha(t_k)\| \\
&\qquad -\sum_{j=2}^{k-1}2^{-j}\left(\|\alpha(t_j)\|+\|\alpha(t_{j-1})\|\right)-\|y\| \\
> &\, 2^{-k-2}\|\alpha(t_k)\|-2^{-k-1}\|\alpha(b)\|-2^{-k}\|\alpha(t_{k-1})\|-\sum_{j=2}^{k-1}2^{-j}\left(2\|\alpha(t_{k-1})\|\right)-\|y\| \\
> &\, 2^{2^k-k-2}\|\alpha(t_{k-1})\|-2^{-k-1}\|\alpha(b)\|-2^{-k}\|\alpha(t_{k-1})\|-2\|\alpha(t_{k-1})\|-\|y\| \\
\geqslant &\, 2^{2^k-k-3}\|\alpha(t_{k-1})\|-2^{-k-1}\|\alpha(b)\|-\|y\|>1.
\end{aligned}$$

This shows that every $y \in Y$ cannot be the interior integral of f with respect to α, contradicting our assumption that α is an integrator. Therefore α is bounded with $\|\alpha\|_\infty := \sup_{t\in[a,b]} \|\alpha(t)\| < \infty$.

Next we show that each fixed partition induces a bounded linear transformation form $\mathrm{Reg}(X)$ to Y. Let $P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$ and let

$$\mathbf{t}_P^* = \left\{t_j^* \in (t_{j-1}, t_j),\ 1 \leqslant j \leqslant n(P)\right\}.$$

A straightforward application of triangle inequality shows that the map $T_{P,\mathbf{t}_P^*}$ defined by

$$T_{P,\mathbf{t}_P^*}(f) = \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) \qquad \forall\ f \in \mathrm{Reg}(X)$$

is a bounded linear transformation from $\mathrm{Reg}(X)$ to Y (norm $\leqslant 2(n(P))\|\alpha\|_\infty$).

Let $f \in \mathrm{Reg}(X)$. We show that there is an $M_f > 0$ such that

$$\left\|T_{P,\mathbf{t}_P^*}(f)\right\| \leqslant M_f \qquad \forall\ P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right) \in \mathscr{P}[a,b],$$
$$\forall\ t_j^* \in (t_{j-1}, t_j):\ 1 \leqslant j \leqslant n(P).$$

Since α is an integrator, $y = \int_a^b [d\alpha(t)](f(t)) \in Y$ exists. With $\varepsilon = 1$ in the definition of interior integral (†), there is a partition

$$P_0\left(a = s_0 < s_1 < s_2 < \cdots < s_{n(P_0)} = b\right)$$

such that if $P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$ satisfies

$$\left\{s_j:\ 1 \leqslant j \leqslant n(P_0)\right\} \subseteq \{t_i:\ 1 \leqslant i \leqslant n(P)\}$$

and

$$t_j^* \in (t_{j-1}, t_j),\ 1 \leqslant j \leqslant n(P)$$

then

$$\left\|y - \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*))\right\| < 1$$

Let $P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$ be an arbitrary partition, and let

$$\mathbf{t}_P^* = \left\{t_j^* \in (t_{j-1}, t_j):\ 1 \leqslant j \leqslant n(P)\right\}.$$

Let Q be the minimum common refinement of P and P_0, i. e.,

$$Q(a = u_0 < u_1 < \cdots < u_{n(Q)} = b),$$
$$\left\{s_i:\ 1 \leqslant i \leqslant n(P_0)\right\} \cup \left\{t_j:\ 1 \leqslant j \leqslant n(P)\right\} = \left\{u_l:\ 1 \leqslant l \leqslant n(Q)\right\}.$$

Choose $u_l^* \in (u_{l-1}, u_l)$ in such a way that $u_l^* = t_j^*$ if $t_j^* \in (u_{l-1}, u_l)$ for some j, and arbitrary $u_l^* \in (u_{l-1}, u_l)$ if $t_j \notin (u_{l-1}, u_l)$ for all $1 \leqslant j \leqslant n(P_0)$.

Since each u_l is either an s_i or a t_j, and since each

$$[\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) = \left(\sum_{(u_{l-1}, u_l) \subseteq (t_{j-1}, t_j)} [\alpha(u_l) - \alpha(u_{l-1})] \right) (f(t_j^*))$$

after cancellation of like terms (terms involving intervals of the form $(t_{j-1}, t_j) = (s_{i-1}, s_i)$ $= (u_{l-1}, u_l)$ for some i, j, l), all remaining terms in the difference

$$\sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) - \sum_{l=1}^{n(Q)} [\alpha(u_l) - \alpha(u_{l-1})](f(u_l^*))$$

involve at least one s_i of the form $[\alpha(s_i) - \alpha(t_j)](f(t_j^*) - f(u_l^*))$ or other two of its variants. Since each s_i can appear in at most two such terms, there can be at most $2(n(P_0))$ terms of this form. Hence

$$\left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) - \sum_{l=1}^{n(Q)} [\alpha(u_l) - \alpha(u_{l-1})](f(u_l^*)) \right\| \leqslant 2(n(P_0))(2\|\alpha\|_\infty)(2\|f\|).$$

Since Q is a refinement of P_0, we have

$$\begin{aligned} \left\| T_{P,\mathbf{t}_P^*}(f) \right\| &= \left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) \right\| \\ &\leqslant \left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](f(t_j^*)) - \sum_{l=1}^{n(Q)} [\alpha(u_l) - \alpha(u_{l-1})](f(u_l^*)) \right\| \\ &\quad + \left\| \sum_{l=1}^{n(Q)} [\alpha(u_l) - \alpha(u_{l-1})](f(u_l^*)) - y \right\| + \|y\| \\ &< 8(n(P_0)) \|\alpha\|_\infty \|f\| + 1 + \|y\| =: M_f, \end{aligned}$$

which is the desired M_f, independent of the partition P and the choice of sampling points $\mathbf{t}_P^* = \{t_j\}$. Since the foregoing argument holds for each fixed f, by the uniform boundedness principle, there exists an $M > 0$ such that

$$\left\| T_{P,\mathbf{t}_P^*} \right\| \leqslant M \quad \text{for all partitions } P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$$
$$\text{and for all choices of } \mathbf{t}_P^* = \left\{ t_j^* \in (t_{j-1}, t_j) : 1 \leqslant j \leqslant n(P) \right\}.$$

It follows that, for each $f \in \mathrm{Reg}(X)$ and each $\varepsilon > 0$, there are a partition P and a choice of sampling points $\mathbf{t}_P^*$ such that

$$\begin{aligned}\|\widehat{\alpha}(f)\| &= \left\| \int_a^b [d\alpha(t)](f(t)) \right\| \leqslant \left\| \int_a^b [d\alpha(t)](f(t)) - T_{P,\mathbf{t}_P^*}(f) \right\| + \left\| T_{P,\mathbf{t}^*}(f) \right\| \\ &< \varepsilon + \left\| T_{P,\mathbf{t}^*} \right\| \|f\| \leqslant \varepsilon + M\|f\|\end{aligned}$$

Since this is true for each $\varepsilon > 0$, the map

$$\widehat{\alpha} : f \mapsto \int_a^b [d\alpha(t)](f(t)) \qquad \forall \ f \in \mathrm{Reg}(X)$$

is a bounded linear transformation from $\mathrm{Reg}(X)$ to Y with norm $\|\widehat{\alpha}\| \leqslant M$. □

Next we derive from the definitions an explicit formula for the integral of a step function with respect to an integrator. This may be obvious to experts, but we feel that there is a need for a proof.

LEMMA 3. *Let $g : [a,b] \to X$ be a step function and let $\alpha : [a,b] \to \mathfrak{B}(X,Y)$ be an integrator for* $\mathrm{Reg}(X)$. *Then there exist a partition $P_0 (a = u_0 < u_1 < \cdots < u_{n(P_0)} = b)$ and $x_k \in X$, $1 \leqslant k \leqslant n(P)$, such that*

$$\int_a^b [d\alpha(t)](g(t)) = \sum_{k=1}^{n(P_0)} [\alpha(u_k) - \alpha(u_{k-1})]x_k = \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})](g(t_j^*))$$

for all partitions $P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$ satisfying

$$\left\{u_k : 1 \leqslant k \leqslant n(P_0)\right\} \subseteq \left\{t_j : 1 \leqslant j \leqslant n(P)\right\},$$

and for all choices of $t_j^* \in (t_{j-1}, t_j)$, $1 \leqslant j \leqslant n(P)$.

This may be obvious to the experts. Since we are aiming at non-experts, a proof according to the definition is desirable.

Proof. Let $P_0 \left(a = u_0 < u_1 < u_2 < \cdots < u_{n(P_0)} = b\right)$ be the partition associated with the step function g. We show that

$$y := \sum_{j=1}^{n(P_0)} [\alpha(u_j) - \alpha(u_{j-1})]x_j = \int_a^b [d\alpha(t)](g(t)).$$

To that end, let $P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$ satisfy

$$\left\{u_j :\ 1 \leqslant j \leqslant n(P_0)\right\} \subseteq \left\{t_k : 1 \leqslant k \leqslant n(P)\right\}.$$

For each $k = 1,2,\cdots,n(P)$, arbitrarily choose $t_k^* \in (t_{k-1}, t_k)$. Observe that, for each $1 \leqslant j \leqslant n(P_0)$, since

$$\bigcup_{(t_{k-1},t_k)\subseteq(u_{j-1},u_j)} (t_{k-1},t_k] = (u_{j-1},u_j],$$

by cancellations of intermediate terms,

$$\sum_{(t_{k-1},t_k)\subseteq(u_{j-1},u_j)} [\alpha(t_k)-\alpha(t_{k-1})] = \alpha(u_{j-1})-\alpha(u_j).$$

Therefore, since $t_k^* \in (u_{j-1},\, u_j)$ and $g(t_k^*) = x_j$ whenever $(t_{k-1},\, t_k) \subseteq (u_{j-1},\, u_j)$,

$$\begin{aligned}
&\sum_{k=1}^{n(P)} [\alpha(t_k)-\alpha(t_{k-1})](g(t_k^*)) \\
&= \sum_{j=1}^{n(P_0)} \left[\sum_{(t_{k-1},t_k)\subseteq(u_{j-1},u_j)} [\alpha(t_k)-\alpha(t_{k-1})](g(t_k^*)) \right] \\
&= \sum_{j=1}^{n(P_0)} \left[\sum_{(t_{k-1},t_k)\subseteq(u_{j-1},u_j)} [\alpha(t_k)-\alpha(t_{k-1})]x_j \right] \\
&= \sum_{j=1}^{n(P_0)} \left[\left(\sum_{(t_{k-1},t_k)\subseteq(u_{j-1},u_j)} [\alpha(t_k)-\alpha(t_{k-1})] \right) x_j \right] \\
&= \sum_{j=1}^{n(P_0)} [\alpha(u_j)-\alpha(u_{j-1})]x_j = y.
\end{aligned}$$

This shows that the vector y satisfies

$$\left\| y - \sum_{k=1}^{n(P)} [\alpha(t_k)-\alpha(t_{k-1})](g(t_k^*)) \right\| = 0$$

for all partition $P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$ and all choices of sampling points $t_j^* \in (t_{j-1}, t_j)$. Hence the vector y is also the interior integral of g with respect to α. $\square$

This leads to the following useful necessary condition for an operator-valued function to be an integrator.

THEOREM 4. *Let* $\alpha : [a,b] \to \mathfrak{B}(X,Y)$ *be an integrator for* $\mathrm{Reg}(X)$. *Then for all partition* $P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$ *and for all choices of* $x_j \in [X]_1$, $1 \leqslant j \leqslant n(P)$,

$$\left\| \sum_{j=1}^{n(P)} [\alpha(t_j)-\alpha(t_{j-1})]x_j \right\| \leqslant \|\widehat{\alpha}\|.$$

Proof. First note that by Theorem 2 the map, $\widehat{\alpha} : f \mapsto \int_a^b [d\alpha(t)](f(t))$, is a bounded linear transformation from $\mathrm{Reg}(X)$ to Y. To see that $\|\widehat{\alpha}\|$ has the asserted property, let a partition $P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$ and $x_j \in [X]_1$, $1 \leqslant j \leqslant n(P)$, be given. Define

$$g(t) = x_j \text{ for } t \in (t_{j-1}, t_j),\ 1 \leqslant j \leqslant n(P), \text{ and } g(t) = 0 \text{ for all other } t\text{'s}.$$

Then g is a step function and hence is in $\mathrm{Reg}(X)$ with $\|g\| \leqslant 1$. Thus, by Lemma 3,

$$\left\| \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})]x_j \right\| = \left\| \int_a^b [d\alpha(t)](g(t)) \right\| = \|\widehat{\alpha}(g)\| \leqslant \|\widehat{\alpha}\| \, \|g\| \leqslant \|\widehat{\alpha}\|. \qquad \square$$

By virtue of this result, we introduce the following notation. For a given $\alpha : [a,b] \to \mathfrak{B}(X,Y)$ define, the set $[\alpha]_1$ as follows.

$$[\alpha]_1 := \left\{ \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})]x_j : P(a = t_0 < \cdots < t_{n(P)} = b) \in \mathscr{P}, \right.$$
$$\left. x_j \in [X]_1,\ 1 \leqslant j \leqslant n(P) \right\}$$

A function $\alpha : [a,b] \to \mathfrak{B}(X,Y)$ having bounded $[\alpha]_1$ is said to have *bounded semivariation*. The smallest bound for $[\alpha]_1$ will be denoted by $V_s(\alpha) = \sup_{y \in [\alpha]_1} \|y\|$, which is also called the *semivariation* of α.

It follows from Theorem 4 that each integrator α is of bounded semivariation with $V_s(\alpha) \leqslant \|\widehat{\alpha}\|$. The converse is also true. A proof can be found in [6]. We give a more elementary proof of the converse, aiming at non-experts (and meanwhile reducing the complication of the treatment of the generality in [6], buried in many pages of the text) for completeness and self containedness. For our proof, we need the following lemmas.

LEMMA 5. *Let $\alpha : [a,b] \to \mathfrak{B}(X,Y)$ be of bounded semivariation and $t \in [a,b]$. Then*

$$\|\alpha(t) - \alpha(a)\| \leqslant V_s(\alpha).$$

Proof. Observe that

$$\begin{aligned} \|\alpha(t) - \alpha(a)\| &= \sup_{x \in [X]_1} \|[\alpha(t) - \alpha(a)]x\| \\ &= \sup_{x \in [X]_1} \|[\alpha(t) - \alpha(a)]x + [\alpha(b) - \alpha(t)]0\| \leqslant V_s(\alpha). \qquad \square \end{aligned}$$

LEMMA 6. *Let $g,h : [a,b] \to X$ be step functions given by*

$$a = u_0 < u_1 < \cdots < u_{n(g)} = b,\ \ g(t) = x_j \in X,\ \ t \in (u_{j-1}, u_j),\ \ 1 \leqslant j \leqslant n(g),$$

and

$$a = v_0 < v_1 < \cdots < v_{n(h)} = b, \;\; h(t) = w_k \in X, \;\; t \in (v_{k-1}, v_k), \;\; 1 \leqslant k \leqslant n(h).$$

Let $\alpha : [a,b] \to \mathfrak{B}(X,Y)$ *be of bounded semivariation; and let*

$$y_g = \sum_{j=1}^{n(g)} [\alpha(u_j) - \alpha(u_{j-1})]x_j \qquad \text{and} \qquad y_h = \sum_{k=1}^{n(h)} [\alpha(v_k) - \alpha(v_{k-1})]w_k.$$

Then

$$\|y_g - y_h\| \leqslant \|g - h\| V_s(\alpha).$$

Proof. Relabel the elements of the set $\{u_j : 0 \leqslant j \leqslant n(g)\} \cup \{v_k : 0 \leqslant k \leqslant n(h)\}$

as $a = t_0 < t_1 < \cdots < t_N = b$ to eliminate duplicates.

Then, for each $1 \leqslant j \leqslant n(g)$ and $1 \leqslant k \leqslant n(h)$,

$$\bigcup_{(t_{i-1},\, t_i) \subseteq (u_{j-1},\, u_j)} (t_{i-1},\, t_i] = (u_{j-1},\, u_j],$$

and

$$\bigcup_{(t_{i-1},\, t_i) \subseteq (v_{k-1},\, v_k)} (t_{i-1},\, t_i] = (v_{k-1},\, v_k].$$

Thus

$$\sum_{(t_{i-1}, t_i) \subseteq (u_{j-1}, u_j)} [\alpha(t_i) - \alpha(t_{i-1})] = \alpha(u_{j-1}) - \alpha(u_j)$$

and

$$\sum_{(t_{i-1}, t_i) \subseteq (v_{k-1}, v_k)} [\alpha(t_i) - \alpha(t_{i-1})] = \alpha(v_{k-1}) - \alpha(v_k).$$

Choose $t_i^* \in (t_{i-1}, t_i)$. Then $g(t_i^*) = x_j$ whenever $(t_{i-1}, t_i) \subseteq (u_{j-1}, u_j)$, and $h(t_i^*) = w_k$ whenever $(t_{i-1}, t_i) \subseteq (v_{k-1}, v_k)$. Since

$$\left\| \|g - h\|^{-1} (g(t_i^*) - h(t_i^*)) \right\| \leqslant 1, \;\; 1 \leqslant i \leqslant N,$$

it follows that

$$\begin{aligned} \|y_g - y_h\| &= \left\| \sum_{j=1}^{n(g)} [\alpha(u_j) - \alpha(u_{j-1})]x_j - \sum_{k=1}^{n(h)} [\alpha(v_k) - \alpha(v_{k-1})]w_k \right\| \\ &= \left\| \sum_{j=1}^{n(g)} \left[\sum_{(t_{i-1}, t_i) \subseteq (u_{j-1}, u_j)} [\alpha(t_i) - \alpha(t_{i-1})] \right] x_j \right. \\ &\qquad \left. - \sum_{k=1}^{n(h)} \left[\sum_{(t_{i-1}, t_i) \subseteq (v_{k-1}, v_k)} [\alpha(t_i) - \alpha(t_{i-1})] \right] w_k \right\| \end{aligned}$$

$$
\begin{aligned}
&= \Bigg\| \sum_{j=1}^{n(g)} \Bigg[\sum_{(t_{i-1},t_i)\subseteq(u_{j-1},u_j)} [\alpha(t_i)-\alpha(t_{i-1})] \Bigg] (g(t_i^*)) \\
&\qquad\qquad - \sum_{k=1}^{n(h)} \Bigg[\sum_{(t_{i-1},t_i)\subseteq(v_{k-1},v_k)} [\alpha(t_i)-\alpha(t_{i-1})] \Bigg] (h(t_i^*)) \Bigg\| \\
&= \Bigg\| \sum_{j=1}^{n(g)} \Bigg[\sum_{(t_{i-1},t_i)\subseteq(u_{j-1},u_j)} [\alpha(t_i)-\alpha(t_{i-1})](g(t_i^*)) \Bigg] \\
&\qquad\qquad - \sum_{k=1}^{n(h)} \Bigg[\sum_{(t_{i-1},t_i)\subseteq(v_{k-1},v_k)} [\alpha(t_i)-\alpha(t_{i-1})](h(t_i^*)) \Bigg] \Bigg\| \\
&= \Bigg\| \sum_{i=1}^{N} [\alpha(t_i)-\alpha(t_{i-1})](g(t_i^*)) - \sum_{i=1}^{N} [\alpha(t_i)-\alpha(t_{i-1})](h(t_i^*)) \Bigg\| \\
&= \Bigg\| \sum_{i=1}^{N} [\alpha(t_i)-\alpha(t_{i-1})](g(t_i^*)-h(t_i^*)) \Bigg\| \\
&= \Bigg\| \sum_{i=1}^{N} [\alpha(t_i)-\alpha(t_{i-1})] \left[\|g-h\|^{-1} (g(t_i^*)-h(t_i^*)) \right] \Bigg\| \, \|g-h\| \\
&\leqslant V_s(\alpha)\, \|g-h\|. \quad \square
\end{aligned}
$$

THEOREM 7. *Let* $\alpha : [a,b] \to \mathfrak{B}(X,Y)$ *be a function of bounded semivariation. Then* α *is an integrator for* $\operatorname{Reg}(X)$.

Proof. Let $\varepsilon > 0$. By Theorem 1 there exists a sequence $\{g_k\}_{k\in\mathbb{N}}$ of step functions $g_k : [a,b] \to X$ such that

$$
\|f - g_k\| = \sup_{a\leqslant t\leqslant b} \|f(t) - g_k(t)\|_X \to 0 \quad \text{as } k \to \infty.
$$

For each $k \in \mathbb{N}$, let $y_k \in Y$ be the vector associated with g_k by Lemma 6. The lemma also gives us

$$
\|y_k - y_l\|_Y \leqslant V_s(\alpha)\, \|g_k - g_l\| \leqslant V_s(\alpha) \left[\|g_k - f\| + \|f - g_l\| \right] \to 0 \text{ as } k,\ l \to \infty.
$$

Thus $\{y_k\}$ is a Cauchy sequence in Y. By the completeness of Y, there is a $y \in Y$ such that $\|y - y_k\|_Y \to 0$ as $k \to \infty$. We show that y has the property (†), i.e., is the integral of f. Let $\varepsilon > 0$ be given. Then there is an $N \in \mathbb{N}$ such that

$$
\|y - y_k\|_Y < \frac{\varepsilon}{2} \qquad \text{and} \qquad \|f - g_k\| < \min\left\{1,\ \frac{\varepsilon}{2[V_s(\alpha)+1]}\right\} \quad \forall\ k \geqslant N.
$$

Let $g = g_N$. Since g is a step function, there exist

$$
a = u_0 < u_1 < u_2 < \cdots < u_{n(g)} = b, \qquad x_j \in X,\ \ 1 \leqslant j \leqslant n(g)
$$

such that $g(t) = x_j$, for $t \in (u_{j-1}, u_j)$, $1 \leqslant j \leqslant n(g)$. Let P_0 be the partition defined by the division points u_j, i.e., $P_0(a = u_0 < u_1 < \cdots < u_{n(g)} = b)$. Let

$$P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$$

be a partition that satisfies

$$\left\{u_j :\ 1 \leqslant j \leqslant n(g)\right\} \subseteq \left\{t_k :\ 1 \leqslant k \leqslant n(P)\right\}.$$

Let $t_j^* \in (t_{j-1}, t_j)$, $1 \leqslant j \leqslant n(g)$, be arbitrarily chosen. Then since $g = g_N$ is a step function, by Lemma 3

$$\sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})](g(t_i^*)) = \sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})](g_N(t_i^*)) = y_N.$$

Since $\left\| \|g - f\|^{-1} (g(t_i^*) - f(t_i^*)) \right\| \leqslant 1$, $1 \leqslant i \leqslant n(P)$, we have

$$\begin{aligned}
&\left\| y - \sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})](f(t_i^*)) \right\| \\
&\leqslant \left\| y - \sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})] g(t_i^*) \right\| + \left\| \sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})](g(t_i^*) - f(t_i^*)) \right\| \\
&= \|y - y_N\| + \left\| \sum_{i=1}^{n(P)} [\alpha(t_i) - \alpha(t_{i-1})] \left[\|g - f\|^{-1} (g(t_i^*) - f(t_i^*)) \right] \right\| \|g - f\| \\
&< \frac{\varepsilon}{2} + V_s(\alpha) \|g - f\| < \frac{\varepsilon}{2} + \frac{\varepsilon V_s(\alpha)}{2[V_s(\alpha) + 1]} < \varepsilon. \quad \square
\end{aligned}$$

Combining Theorems 7 and 4, we have the following.

COROLLARY 8.

1. *A function* $\alpha : [a, b] \to \mathfrak{B}(X, Y)$ *is an integrator for* $\mathrm{Reg}(X)$ *if, and only if,* α *is of bounded semivariation.*

2. *For each integrator* α *for* $\mathrm{Reg}(X)$, $V_s(\alpha) = \|\widehat{\alpha}\|$.

Proof. Part (1) follows directly from Theorems 7 and 4.

For part (2), we have already noted above that $V_s(\alpha) \leqslant \|\widehat{\alpha}\|$. For the opposite inequality, let $f \in \mathrm{Reg}(X)$, and let $\varepsilon > 0$. Let P_0 and all s_j be as in (†). Choose any $s_j^* \in (s_{j-1}, s_j)$, for $1 \leqslant j \leqslant n(P_0)$. Then, since $\left\| \|f\|^{-1} f(s_j^*) \right\| \leqslant 1$ for all $1 \leqslant j \leqslant n(P_0)$, we have

$$\begin{aligned}\|\widehat{\alpha}(f)\| &\leqslant \left\| \int_a^b [d\alpha(t)](f(t)) - \left[\sum_{j=1}^{n(P_0)} [\alpha(s_j)-\alpha(s_{j-1})](f(s_j^*)) \right] \right\| \\ &\qquad + \left\| \sum_{j=1}^{n(P_0)} [\alpha(s_j)-\alpha(s_{j-1})](f(s_j^*)) \right\| \\ &< \varepsilon + \left\| \sum_{j=1}^{n(P_0)} [\alpha(s_j)-\alpha(s_{j-1})](\|f\|^{-1} f(s_j^*)) \right\| \cdot \|f\| \leqslant \varepsilon + V_s(\alpha)\,\|f\|\end{aligned}$$

Since ε is arbitrary, we have $\|\widehat{\alpha}\| \leqslant V_s(\alpha)$, and hence equality follows. □

4. Compactness of integrators

We first consider the following example. Let $X = Y = \ell^2$ (the square summable Hilbert sequence space), and $[a,b] = [0,1]$. For each $j \in \mathbb{N}$, denote by e_j the j^{th} standard basis vector of ℓ^2, and E_j the orthogonal projection of ℓ^2 onto the span of e_j (i.e, $E_j = e_j \otimes e_j$, or $E_j(x) = \langle x, e_j \rangle e_j$ for all $x \in \ell^2$). Define $\alpha : [0,1] \to \mathfrak{B}(\ell^2)$ by $\alpha(t) = \frac{1}{\sqrt{n}} E_n$ for all $t \in (2^{-n}, 2^{-n+1}]$, $n \in \mathbb{N}$, and $\alpha(0) = 0$. With $N \geqslant 2$, and with the partition $P(0 < 2^{-N} < 2^{-N+1} < \cdots < 1)$ and $x_1 = x_{N+1} = 0$, $x_{j+1} = e_{N-j}$, $1 \leqslant j \leqslant N-1$, we have

$$\begin{aligned}&\left\| [\alpha(2^{-N}) - \alpha(0)]x_1 + \sum_{j=1}^{N} [\alpha(2^{-N+j}) - \alpha(2^{-N+j-1})]x_{j+1} \right\| \\ &= \left\| \sum_{j=1}^{N-1} [\alpha(2^{-N+j}) - \alpha(2^{-N+j-1})]e_{N-j} \right\| = \left\| \sum_{j=1}^{N-1} (A_{N-j+1} - A_{N-j})e_{N-j} \right\| \\ &= \left\| \sum_{j=1}^{N-1} A_{N-j}e_{N-j} \right\| = \left\| \sum_{j=1}^{N-1} \frac{1}{\sqrt{N-j}} e_{N-j} \right\| = \left[\sum_{j=1}^{N-1} \frac{1}{N-j} \right]^{1/2} \to \infty \qquad \text{as} \quad N \to \infty.\end{aligned}$$

Thus α is not an integrator by Theorem 4, though each $\alpha(t)$ is of rank one and hence compact.

Recall that a subset of a metric space is said to be *totally bounded* if, for every $\varepsilon > 0$, it can be covered by a finite number of ε-balls. A subset of a metric space is totally bounded iff it has a compact closure. A bounded linear transformation K from a Banach space X to a Banach space Y is compact if the image of the unit ball of X under K has a compact closure. Thus K is compact iff $K([X]_1)$ (or the image under K of each bounded subset of X) is totally bounded.

THEOREM 9. *Let $\alpha : [a,b] \to \mathfrak{B}(X,Y)$ be an integrator for* $\operatorname{Reg}(X)$ *that vanishes at a (i.e., $\alpha(a) = 0$). Then the following conditions on α are equivalent.*

1. *$\widehat{\alpha}$ is compact,*
2. *There is a compact operator K from a Banach space Z to Y such that $[\alpha]_1 \subseteq K([Z]_r)$ for some $r > 0$.*
3. *There is a one-to-one compact operator K from a Banach space Z to Y such that $K^{-1}(\alpha(t))$ is a bounded operator from X to Z for all $t \in [a,b]$ and $K^{-1} \circ \alpha$ is an integrator.*
4. *The set $[\alpha]_1$ is a totally bounded subset of Y.*

A modification of the example preceding the statement of the theorem above can also serve as examples for (2)–(4).

Define $\beta(t) = n^{-2} E_n$ for $t \in (2^{-n-1}, 2^n]$, $n \in \mathbb{N}$, and $\beta(0) = 0$. Let $P(a = t_0 < \cdots < t_{N(P)})$ be a partition of $[0,1]$, and let $x_j \in [X]_1$, $1 \leqslant j \leqslant N(P)$.The nonzero terms in the sum in $[\alpha]_1$ are those j that satisfy $t_j > 2^{-k} \geqslant 2^{-m} \geqslant t_{j-1}$ for some $k \geqslant m$. Thus

$$\sum_{j=1}^{n(P)} [\beta(t_j) - \beta(t_{j-1})]x_j = \sum_{\substack{1 \leqslant j \leqslant n(P) \\ t_j > 2^{-k} \geqslant 2^{-m} \geqslant t_{j-1}}} \left[k^{-2} \langle x_j, e_k \rangle e_k - m^{-2} \langle x_j, e_m \rangle e_m \right]$$

Notice that e_k can at most appear as $(k+1)^{-2} \langle x_{j+1}, e_k \rangle e_k$ in the term preceding it. So the combined e_k terms has coefficient

$$k^{-2} \langle x_j, e_k \rangle - (k+1)^{-2} \langle x_{j+1}, e_k \rangle = k^{-1} \left(k^{-1} \langle x_j, e_k \rangle - k(k+1)^{-2} \langle x_{j+1}, e_k \rangle \right) = k^{-1} \left(k^{-1} a_k \right)$$

for some $|a_k| \leqslant 2$, since $x_j, x_{j+1} \in [X]_1$. Similar expression holds for e_m. Since $\sum_{k=1}^{\infty} k^{-1} e_k \in \ell^2$ (i.e., the sequence $\left\{ n^{-1} \right\}_{n \in \mathbb{N}}$ is in ℓ^2), the whole sum is in the range of the compact operator $K := \sum_{\nu=1}^{\infty} \nu^{-1} E_\nu$. So β satisfies condition (2) of the this theorem with $[\beta]_1 \subseteq K([\ell^2]_2)$.

Let $L = \sum_{\nu=1}^{\infty} \nu^{-1/2} E_\nu$, and let $\gamma = L^{-1} \circ \beta$, we see that

$$\gamma(t) = L^{-1}(\beta(t)) = \frac{1}{n^{3/2}} E_n \text{ if } t \in (2^{-n}, 2^{-n+1}] \text{ for some } n, \text{ and } \gamma(0) = L^{-1}(\beta(0)) = 0.$$

Arguments similar to that used for β show that

$$[\gamma]_1 = [L^{-1} \circ \beta]_1 \subseteq L([X]_2) \qquad \text{a bounded set.}$$

Thus β satisfies condition (3). Since L is compact, $L([X]_2)$ is totally bounded, so that γ also satisfies condition (4)

Proof. [(1) $\Rightarrow$ (2)] Suppose $\widehat{\alpha}$ is compact. We show that α satisfies (2) with $K = \widehat{\alpha}$ and $Z = \mathrm{Reg}(X)$. Each element of $[\alpha]_1$ corresponds to a partition

$$P(a = t_0 < \cdots < t_{n(P)} = b) \in \mathscr{P}[a,b]$$

and a finite collection of vectors $x_j \in [X]_1$, $1 \leqslant j \leqslant n(P)$.

Then the function g defined by $g(t) = x_j$ for $t \in [t_{j-1}, t_j)$, and $g(b) = 0$, is a step function and hence belongs to $[\mathrm{Reg}(X)]_1$. Thus, by Lemma 3,

$$\sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})]x_j = \widehat{\alpha}(g) \in \widehat{\alpha}([\mathrm{Reg}(X)]_1).$$

Thus $[\alpha]_1$ is contained in $\widehat{\alpha}([\mathrm{Reg}(X)]_1) = K([Z]_1)$.

[(2) $\Rightarrow$ (3)] Suppose $[\alpha]_1 \subseteq K([Z]_r)$ for some compact operator K from a Banach space Z to Y, and for some $r > 0$. Considering the compact operator $\tilde{K}$ induced on the quotient space $Z/\ker K$, we may assume that K is one-to-one. For $t \in [a,b]$ and $x \in [X]_1$, since the function

$$g_{t,x}(s) = \begin{cases} x & \text{for } s \in [a,t] \\ 0 & \text{for } s \in (t,b] \end{cases}$$

is in $[\mathrm{Reg}(X)]_1$, we have

$$\begin{aligned} [\alpha(t)]x =& [\alpha(t) - \alpha(a)]x = \int_a^b [d\alpha(s)](g_{t,x}(s)) \\ =& \widehat{\alpha}(g_{t,x}) \in \widehat{\alpha}([\mathrm{Reg}(X)]_1) \subseteq K([Z]_1). \end{aligned}$$

Thus $[\alpha(t)](X) \subseteq K(Z)$ for all $t \in [a,b]$. Hence $K^{-1}[\alpha(t)]$ is a closed operator from the Banach space X to Z. Thus each $K^{-1}[\alpha(t)]$ $(t \in [a,b])$ is a bounded operator from X to Z. Furthermore $\beta := K^{-1} \circ \alpha : [a,b] \to \mathfrak{B}(X,Z)$ satisfies

$$\begin{aligned} [\beta]_1 =& \left\{ \sum_{j=1}^{n(P)} [\beta(t_j) - \beta(t_{j-1})]x_j : P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right), \right. \\ & \left. x_j \in [X]_1,\ 1 \leqslant j \leqslant n(P) \right\} \\ =& \left\{ \sum_{j=1}^{n(P)} [K^{-1}(\alpha(t_j)) - K^{-1}(\alpha(t_{j-1}))]x_j : P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right), \right. \\ & \left. x_j \in [X]_1,\ 1 \leqslant j \leqslant n(P) \right\} \\ =& K^{-1}\left(\left\{ \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})]x_j : P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right), \right. \right. \\ & \left. \left. x_j \in [X]_1,\ 1 \leqslant j \leqslant n(P) \right\} \right) \\ =& = K^{-1}([\alpha]_1) \subseteq [Z]_r. \end{aligned}$$

Therefore β is an integrator.

[(3) $\Rightarrow$ (4)] Suppose $\beta := K^{-1} \circ \alpha$ defines an integrator for some compact operator K from Z to Y. Then $[\beta]_1 = K^{-1}([\alpha]_1)$ is a bounded subset of Z. Thus $[\alpha]_1 = K(K^{-1}([\alpha]_1)) = K([\beta]_1)$ is a totally bounded set (see the discussion preceding the statement of this theorem).

[(4) $\Rightarrow$ (1)] Suppose $[\alpha]_1$ is totally bounded. We show that $\widehat{\alpha}([\mathrm{Reg}(X)]_1)$ is contained in the closure of $[\alpha]_1$. Let $f \in [\mathrm{Reg}(X)]_1$, and $\varepsilon > 0$. Then, by definitions of $\widehat{\alpha}$ and the integral, there is a partition $P\left(a = t_0 < t_1 < t_2 < \cdots < t_{n(P)} = b\right)$ such that for all choices of $t_j^* \in (t_{j-1}, t_j)$, $1 \leqslant j \leqslant n(P)$, we have

$$\left\| \widehat{\alpha}(f) - \sum_{j=1}^{n(P)} [\alpha(t_j) - \alpha(t_{j-1})] f(t_j^*) \right\| < \varepsilon.$$

Since $f \in [\mathrm{Reg}(X)]_1$, each $f(t_j^*) \in [X]_1$, and hence the sum in the preceding inequality is in $[\alpha]_1$. As ε is arbitrary, $\widehat{\alpha}(f) \in \overline{[\alpha]}_1$, that is $\widehat{\alpha}([\mathrm{Reg}(X)]_1) \subseteq \overline{[\alpha]}_1$, the closure of the totally bounded set $[\alpha]_1$. Thus $\widehat{\alpha}$ is a compact linear transformation. $\square$

Acknowledgement. The authors gratefully acknowledge the referee for careful reading of the paper and pointing out the result in Hönig, which we missed, and for suggestions leading to significant improvements enhancements of the exposition. T. Wootijirattikal gratefully acknowledges the financial support from Thailand Research Fund.

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(Received December 5, 2017)

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