

## ALGEBRA-VALUED G-FRAMES IN HILBERT $C^*$ -MODULES

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*Abstract.* In this paper, we consider the notion of algebra-valued G-frame as a special case of G-frame in a Hilbert  $C^*$ -module. It is shown that every Hilbert module over a commutative  $C^*$ -algebra  $A$  admits an algebra-valued G-frame iff  $A$  is a  $C^*$ -algebra of compact operators.

### 1. Introduction

M. Frank and D. R. Larson [3] generalized the classical frame theory in Hilbert spaces to Hilbert  $C^*$ -modules. They concluded from Kasparov's stabilization theorem that every countably (finitely) generated Hilbert  $C^*$ -module over a unital  $C^*$ -algebra admits a frame. However, as it is asked in Problem 8.1 of [3], the interesting open question is for which kind of  $C^*$ -algebra  $A$ , every Hilbert  $A$ -module admits a frame.

In 2010, Li characterized a commutative unital  $C^*$ -algebra that every Hilbert  $C^*$ -module over it has a frame as a finite dimensional  $C^*$ -algebra [6]. Li's result for non-unital commutative  $C^*$ -algebras is stated in [1] in the following way.

**THEOREM 1.1.** [1, Theorem. 1.4] *Let  $A$  be a commutative  $C^*$ -algebra. Then  $A$  is a  $C^*$ -algebra of compact operators (equivalently, it has discrete spectrum) if and only if every Hilbert  $A$ -module has a frame.*

As a generalization of frames, Sun in 2006 introduced G-frames in Hilbert spaces [8]. Also, this concept has been generalized to Hilbert  $C^*$ -modules in [5]. In this paper, we define the notion of algebra-valued G-frame in a Hilbert  $C^*$ -module, as a special case of G-frame.

We investigate the existence problem for algebra-valued G-frames in Hilbert  $C^*$ -modules. In fact, we use a generalization of Serre-Swan theorem [4] and prove that for  $A$  being a commutative  $C^*$ -algebra, every Hilbert  $A$ -module admits an algebra-valued G-frame iff  $A$  is a  $C^*$ -algebra of compact operators. In particular, for  $A$  being a unital commutative  $C^*$ -algebra, every Hilbert  $A$ -module admits an algebra-valued G-frame iff  $A$  is finite dimensional.

Assume that  $A$  is a  $C^*$ -algebra and  $X, Y$  are Hilbert  $A$ -modules. The family of all bounded  $A$ -linear maps from  $X$  into  $Y$  is denoted by  $\text{End}(X, Y)$ . Also,  $I$  is an arbitrary indexing set.

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DEFINITION 1.2. Let  $\{Y_i : i \in I\}$  be a family of Hilbert  $A$ -modules. A family  $\{\Lambda_i \in \text{End}(X, Y_i) : i \in I\}$  is called a  $G$ -frame for  $X$  with respect to  $\{Y_i : i \in I\}$ , if there exist constants  $0 < C \leq D < \infty$  such that for every  $x \in X$ ,

$$C\langle x, x \rangle \leq \sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle \leq D\langle x, x \rangle, \tag{1.1}$$

where, by using the standard isometric embedding of  $A$  into its universal enveloping von Neumann algebra  $A^{**}$ , the value  $\sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle$  is the limit of the increasingly ordered net of its finite partial sums with respect to the ultraweak topology on  $A^{**}$ .

We remark that some authors use a slightly different definition so that each operator  $\Lambda_i$  is adjointable.

Obviously, every Hilbert  $A$ -module  $X$  has a  $G$ -frame. Indeed, one can consider the identity map on  $X$ , where  $I$  is a singleton set and  $Y_i = X$ , for  $i \in I$ . However, when we envisage a frame  $\{h_i\}_{i \in I}$  in a Hilbert space  $H$  as a  $G$ -frame, we consider each  $h_i$  as a member of  $H^*$ , the dual of  $H$ . Due to this, algebra-valued  $G$ -frame can be a proper generalization of frame in Hilbert  $C^*$ -modules.

DEFINITION 1.3. A family  $\{\Lambda_i \in \text{End}(X, A) : i \in I\}$  that satisfies the properties of a  $G$ -frame is called algebra-valued  $G$ -frame.

One can state an analogue of Proposition 3.1 in [6] for  $G$ -frames as follows.

PROPOSITION 1.4. A family  $\{(\Lambda_i, Y_i) : i \in I\}$  is a  $G$ -frame for  $X$ , with  $G$ -frame bounds  $C$  and  $D$  if and only if

$$C\varphi(\langle x, x \rangle) \leq \sum_{i \in I} \varphi(\langle \Lambda_i(x), \Lambda_i(x) \rangle) \leq D\varphi(\langle x, x \rangle), \tag{1.2}$$

for any  $x \in X$  and any state  $\varphi$  of  $A$ .

## 2. Hilbert modules over commutative $C^*$ -algebras

A generalization of Serre-Swan theorem states that the category of continuous fields of Hilbert spaces over a locally compact Hausdorff space  $Z$  is equivalent to the category of Hilbert  $C^*$ -modules over the commutative  $C^*$ -algebra  $A = C_0(Z)$  [4, Theorem. 4.8.]. We have applied this fact to show that every Hilbert module over a commutative  $C^*$ -algebra  $A$  admits an algebra-valued  $G$ -frame iff  $A$  is a  $C^*$ -algebra of compact operators.

DEFINITION 2.1. Let  $Z$  be a locally compact Hausdorff space. Consider  $((H_z)_{z \in Z}, \Gamma)$ , where  $(H_z)_{z \in Z}$  is a family of Hilbert spaces and  $\Gamma$  is a subset of  $\prod_{z \in Z} H_z$ . Also, we set

$$C_0 - \prod_{z \in Z} H_z = \{x \in \prod_{z \in Z} H_z : [z \mapsto \|x(z)\|] \in C_0(Z)\}.$$

The pair  $((H_z)_{z \in Z}, \Gamma)$  satisfying the following properties is said to be a continuous field of Hilbert spaces.

- 1)  $\Gamma$  is a linear subspace of  $C_0 - \prod_{z \in Z} H_z$ .
- 2) The set  $\{x(z) : x \in \Gamma\}$  equals to  $H_z$ , for every  $z \in Z$ .
- 3) If  $x \in C_0 - \prod_{z \in Z} H_z$  and for every  $z \in Z$  and every  $\varepsilon > 0$  there is a  $x' \in \Gamma$  such that  $\|x(s) - x'(s)\| < \varepsilon$  in some neighborhood of  $z$ , then  $x \in \Gamma$ .

The space  $\mathcal{H} = \prod_{z \in Z} H_z$  is called the total space.

REMARK 2.2. a) Note that the function  $z \mapsto \langle x(z), y(z) \rangle$  is an element of  $C_0(Z)$ , for every  $x, y \in \Gamma$ . Also, if the topological space  $Z$  is discrete, then  $\Gamma = C_0 - \prod_{z \in Z} H_z$  [2].

b) A morphism  $\psi : ((H_z)_{z \in Z}, \Gamma) \longrightarrow ((K_z)_{z \in Z}, \Gamma')$  of continuous fields of Hilbert spaces is a family of linear maps  $\{\psi_z : H_z \longrightarrow K_z : z \in Z\}$  such that the induced map  $\psi : \mathcal{H} \longrightarrow \mathcal{K}$  on the total spaces satisfies  $\{\psi \circ x : x \in \Gamma\} \subseteq \Gamma'$  and also the map  $z \mapsto \|\psi_z\|$  is locally bounded. By [4, Proposition 4.7.],  $\Gamma$  has a structure of Hilbert  $C_0(Z)$ -module with pointwise multiplication and inner product

$$\langle x, y \rangle(z) = \langle x(z), y(z) \rangle \quad (x, y \in \Gamma, z \in Z).$$

Indeed, the category of Hilbert  $C_0(Z)$ -modules is equivalent to the category of continuous fields of Hilbert spaces. In particular, if  $((H_z)_{z \in Z}, \Gamma)$  and  $((K_z)_{z \in Z}, \Gamma')$  are the corresponding continuous fields of Hilbert spaces to Hilbert  $C_0(Z)$ -modules  $X$  and  $Y$ , then for each  $\Lambda$  in  $End(X, Y)$ , the map  $\Lambda_z : H_z \longrightarrow K_z$  defined by  $\Lambda_z(x(z)) = (\Lambda(x))(z)$  is a well-defined bounded linear operator, for every  $z \in Z$  [4].

c) If we consider  $A = C_0(Z)$  as a Hilbert  $A$ -module, in the natural way, then the corresponding continuous field of Hilbert spaces to Hilbert  $A$ -module  $A$  is  $((\mathbb{C}_z)_{z \in Z}, \Gamma_A)$ , where  $\mathbb{C}_z = \mathbb{C}$ , for every  $z \in Z$  and  $\Gamma_A = \{(f(z))_{z \in Z} : f \in C_0(Z)\}$ . In particular, when  $Z$  is discrete then  $\Gamma_A = C_0 - \prod_{z \in Z} \mathbb{C}_z$ .

THEOREM 2.3. *If  $Z$  is a discrete topological space, then every Hilbert  $C_0(Z)$ -module admits a algebra-valued G-frame.*

*Proof.* Let  $X$  be a Hilbert  $C^*$ -module over a commutative  $C^*$ -algebra  $A = C_0(Z)$ , where  $Z$  is a discrete topological space. There is a continuous field of Hilbert spaces  $((H_z)_{z \in Z}, \Gamma)$  that  $X$  is of the form  $\Gamma$ . Since  $Z$  is discrete, then  $\Gamma = C_0 - \prod_{z \in Z} H_z$ , by part (a) of Remark 2.2.

Let  $\{f_i^z : i \in I_z\}$  be an orthonormal basis for  $H_z$  and  $\mathbb{C}_z = \mathbb{C}$ , for every  $z \in Z$ . For each  $i \in I = \cup_{z \in Z} I_z$ , we define  $\Lambda_i : \Gamma \longrightarrow C_0 - \prod_{z \in Z} \mathbb{C}_z$  by  $\Lambda_i((x_z)_{z \in Z}) = (\lambda_{i,z})_{z \in Z}$ , where  $x_z \in H_z$  and  $\lambda_{i,z} = \langle x_z, f_i^z \rangle$  if  $i \in I_z$  and  $\lambda_{i,z} = 0$  otherwise. Clearly, for every  $x = (x_z)_{z \in Z} \in \Gamma$ , we have

$$\langle x, x \rangle(z) = \langle x_z, x_z \rangle = \sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle(z).$$

Hence,  $\{\Lambda_i\}_{i \in I}$  is a G-frame for  $\Gamma$ . On the other hand, by part (c) of Remark 2.2, there is an  $A$ -module isomorphism  $\Psi$  from  $C_0 - \prod_{z \in Z} \mathbb{C}_z$  onto  $A$ . Obviously,  $\{\Psi \circ \Lambda_i\}_{i \in I}$  is an algebra-valued G-frame for  $\Gamma = X$ .  $\square$

The following proposition is a generalization of [6, Proposition 2.4].

PROPOSITION 2.4. [1, Proposition 1.3] *Let  $Z$  be an infinite non-discrete locally compact Hausdorff space. Then there exist a continuous field of Hilbert spaces  $((H_z)_{z \in Z}, \Gamma)$  over  $Z$ , a countable subset  $W \subseteq Z$  and a point  $z_\infty \in \overline{W} \setminus W$  that  $H_z$  is separable for every  $z \in W$  and  $H_{z_\infty}$  is non-separable.*

THEOREM 2.5. *Let  $Z$  be an infinite non-discrete locally compact Hausdorff space. Then there is a Hilbert  $C_0(Z)$ -module that admits no algebra-valued G-frame.*

*Proof.* With the notations in Proposition 2.4, we show that the Hilbert  $C_0(Z)$ -module  $\Gamma$  admits no algebra-valued G-frames. To get a contradiction, assume that  $A = C_0(Z)$  and  $\{\Lambda_i : \Gamma \rightarrow A : i \in I\}$  is an algebra-valued G-frame for  $\Gamma$ , with bounds  $C$  and  $D$ . Hence, for every  $x \in \Gamma$  and every  $z \in Z$ ,

$$C\|x(z)\|^2 \leq \sum_{i \in I} \langle \Lambda_i(x), \Lambda_i(x) \rangle(z) \leq D\|x(z)\|^2.$$

On the other hand, the Hilbert  $A$ -module  $A$  is isomorphic to  $\Gamma_A$ . Hence, by part (b) of Remark 2.2, for every  $i \in I$ ,  $\Lambda_i$  corresponds to  $\{\Lambda_{i,z} : H_z \rightarrow \mathbb{C}_z : z \in Z\}$ . By Riesz representation theorem, for every  $z \in Z$  we can find a subset  $\{f_i^z : i \in I\}$  of  $H_z$  such that for every  $x \in \Gamma$ ,

$$\Lambda_{i,z}(x(z)) = \langle x(z), f_i^z \rangle. \tag{2.1}$$

Therefore, for every  $x \in \Gamma$ ,

$$C\|x(z)\|^2 \leq \sum_{i \in I} \langle \Lambda_{i,z}(x(z)), \Lambda_{i,z}(x(z)) \rangle \leq D\|x(z)\|^2.$$

By Equation 2.1 and Axiom 2 of Definition 2.1, for every  $z \in Z$  and every  $w \in H_z$ ,

$$C\langle w, w \rangle \leq \sum_{i \in I} \langle w, f_i^z \rangle \langle f_i^z, w \rangle \leq D\langle w, w \rangle. \tag{2.2}$$

The remaining part of the proof is the same as the proof of Lemma 3.2 in [6]. In fact, for each  $z \in Z$ , we first choose an orthonormal basis  $S_z$  for  $H_z$ . By (2.2), for every  $w \in S_z$ ,  $F_w = \{i \in I : \langle w, f_i^z \rangle \neq 0\}$  is countable. For every  $z \in W$ ,  $S_z$  is countable, so  $F_z = \{i \in I : f_i^z \neq 0\}$ , which is equal to  $\bigcup_{w \in S_z} F_w$ , is countable. Hence,  $F = \bigcup_{z \in W} F_z$  is countable and also for every  $i \in I \setminus F$  and every  $z \in W$ ,  $f_i^z = 0$ .

On the other hand, by Remark 2.2, the map  $z \mapsto \langle f_i^z, f_i^z \rangle = \Lambda_{i,z}(f_i^z)$  is continuous. Hence, for every  $i \in I \setminus F$ ,  $f_i^{z_\infty} = 0$ , because  $z_\infty \in \overline{W}$ .

Since  $H_{z_\infty}$  is nonseparable, there is a non-zero  $w \in H_{z_\infty}$  that is orthogonal to  $f_i^{z_\infty}$ , for every  $i \in F$ . Therefore, for every  $i \in I$ , we have  $\langle f_i^{z_\infty}, w \rangle = 0$ . By (2.2),  $w$  is equal to zero, that is a contradiction.  $\square$

COROLLARY 2.6. *Every Hilbert  $C^*$ -module over a commutative  $C^*$ -algebra  $A$  admits an algebra-valued G-frame iff  $A$  is  $C^*$ -algebra of compact operators.*

*Proof.* By Theorems 2.3 and 2.5, every Hilbert  $C^*$ -module over a commutative  $C^*$ -algebra  $A = C_0(Z)$  admits an algebra-valued G-frame iff  $Z$  is discrete. On the other hand,  $C_0(Z)$  is a  $C^*$ -algebra of compact operators iff  $Z$  is discrete [2, 4.7.20].  $\square$

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