

## EXTENSIONS OF HIAI-LIN TYPE EIGENVALUE INEQUALITY

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*Abstract.* In this paper, we prove several extensions of Hiai-Lin type eigenvalue inequality which extends the relative result before.

### 1. Introduction and main results

A capital letter, such as  $T$ , stands for an  $n \times n$  matrix.  $T > 0$  means that  $T$  is a positive definite matrix.  $\lambda_i(T)$  is the  $i$ th largest eigenvalue of  $T$  if  $T$  is Hermitian.

Let  $A\sharp_t B$  stands for the weighted geometric mean. In other words,

$$A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$$

if  $A, B > 0$  and  $t \in [0, 1]$ . Similarly,  $A\sharp_{1-t} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t} A^{\frac{1}{2}}$  if  $A, B > 0$  and  $t \notin [0, 1]$ .

In [1], F. Hiai and M. Lin proved the following eigenvalue inequality.

**THEOREM 1.1.** ([1]) *If  $A, B > 0$ , then*

$$\prod_{i=1}^k \lambda_i(AB) \geq \prod_{i=1}^k \lambda_i((A\sharp_t B)(A\sharp_{1-t} B)), \quad k = 1, 2, \dots, n \quad (1.1)$$

holds for  $t \in [0, 1]$ .

In this paper, we shall show extension of Theorem 1.1 as follows.

**THEOREM 1.2.** *If  $A, B > 0$ , then*

$$\prod_{i=1}^k \lambda_i(AB)^l \geq \prod_{i=1}^k \lambda_i \left( \left( A^{-r-1} (A^{2r+1})^{\frac{1-(1-2t+r)\alpha}{1-t+r}} (A\sharp_{1-t} B) \right) A^{-r-1} \right) \left( A\sharp_{-(1-2t+r)\alpha} B \right) \quad (1.2)$$

holds for  $t \in [0, \frac{1}{2}]$ ,  $\alpha \in [0, 1]$  and  $1 \geq -r > 1-t \geq \frac{1}{2}$ ,  $k = 1, 2, \dots, n$ , where  $l = -\frac{1-2t+r}{1-t+r} \cdot r\alpha$ .

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THEOREM 1.3. *If  $A, B > 0$ , then*

$$\prod_{i=1}^k \lambda_i(AB)^l \geq \prod_{i=1}^k \lambda_i \left( \left( A^{-r-1} \left( A^{2r+1} \natural_{\frac{\alpha r}{1-t+r}} (A \sharp_{1-t} B) \right) A^{-r-1} \right) \left( A \sharp_{-\alpha r} B \right) \right) \quad (1.3)$$

holds for  $t \in [\frac{1}{2}, 1]$ ,  $\alpha \in [0, 1]$  and  $1 \geq -r > 1-t \geq 0$ ,  $k = 1, 2, \dots, n$ , where  $l = -\frac{\alpha r^2}{1-t+r}$ .

In order to prove the main result we list a famous operator inequality – Tanahashi inequality here.

THEOREM 1.4. (Tanahashi inequality [2]) *If  $A \geq B \geq 0$  with  $A > 0$ , then*

$$A^{\frac{p'+2r'}{q'}} \geq (A^{r'} B^{p'} A^{r'})^{\frac{1}{q'}} \quad (1.4)$$

holds for  $0 \leq p' \leq 1$ ,  $0 < q' \leq 1$  and  $-1 \leq 2r' < 0$  satisfying

$$-2r'(1-q') \leq p' \leq q' - 2r'(1-q') \quad (1.5)$$

and

$$\frac{-2r'(1-q') - q'}{1-2q'} \leq p' \leq \frac{-2r'(1-q')}{1-2q'} \quad (\text{when } q' < 1/2). \quad (1.6)$$

REMARK 1.1. If we put  $r' = r/2$ ,  $p' = p$ , and  $q' = \frac{p+r'}{p}$  in (1.4) and (1.5), then we can obtain that  $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ ; If we put  $r' = r/2$ ,  $p' = p$ , and  $q' = \frac{p+r'}{2p'-1+r'}$  in (1.4) and (1.6), then we can obtain that  $A^{2p-1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{2p-1+r}{p+r}}$ . Thus, we can obtain the reformulations of Tanahashi inequality[2]: If  $A \geq B \geq 0$  with  $A > 0$ ,  $r < 0$ , then the following inequalities hold.

$$\begin{cases} \text{Case 1. } A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}, & \text{if } 1 \geq -r > p \geq 0 \text{ with } p \leq \frac{1}{2}; \\ \text{Case 2. } A^{2p-1+r} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{2p-1+r}{p+r}}, & \text{if } 1 \geq -r > p \geq \frac{1}{2}. \end{cases}$$

### 2. Proofs of main results

In this section, we shall prove Theorem 1.2 and Theorem 1.3.

*Proof of Theorem 1.2.* By the well-known antisymmetric tensor power technique, we may need to prove that  $B \leq A^{-1}$  ensures that

$$A^{-r-1} \left( A^{2r+1} \natural_{\frac{(1-2t+r)\alpha}{1-t+r}} (A \sharp_{1-t} B) \right) A^{-r-1} \leq (A \sharp_{-(1-2t+r)\alpha} B)^{-1}. \quad (2.1)$$

$B \leq A^{-1}$  is equivalent to  $A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq A^{-2}$ .

Let  $p = 1-t$  and apply  $A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq A^{-2}$  to Tanahashi inequality(Case 2), we have

$$A^{-2(1-2t+r)} \geq (A^{-r} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{1-t} A^{-r})^{\frac{1-2t+r}{1-t+r}}. \quad (2.2)$$

Because  $-(1-2t+r) \in [0, 1]$  and  $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq A^{-2}$ ,

$$(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-2t+r} \geq A^{-2(1-2t+r)} \quad (2.3)$$

holds by Löwner-Heinz inequality.

Continuing applying Löwner-Heinz inequality for  $\alpha \in [0, 1]$  to (2.2) and (2.3), we have

$$(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{(1-2t+r)\alpha} \geq (A^{-r}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{-r})^{\frac{(1-2t+r)\alpha}{1-t+r}}. \quad (2.4)$$

Notice that

$$\begin{aligned} & A^{-r-1}(A^{2r+1}\natural_{\frac{(1-2t+r)\alpha}{1-t+r}}(A\sharp_{1-t}B))A^{-r-1} \\ &= A^{-r-1}A^{r+\frac{1}{2}}(A^{-r-\frac{1}{2}}A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{\frac{1}{2}}A^{-r-\frac{1}{2}})^{\frac{(1-2t+r)\alpha}{1-t+r}}A^{r+\frac{1}{2}}A^{-r-1} \\ &= A^{-\frac{1}{2}}(A^{-r}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{-r})^{\frac{(1-2t+r)\alpha}{1-t+r}}A^{-\frac{1}{2}} \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} & (A\sharp_{-(1-2t+r)\alpha}B)^{-1} \\ &= (A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-(1-2t+r)\alpha}A^{\frac{1}{2}})^{-1} \\ &= A^{-\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{(1-2t+r)\alpha}A^{-\frac{1}{2}}. \end{aligned} \quad (2.6)$$

Together with (2.4), (2.5) and (2.6), (2.1) holds obviously.  $\square$

*Proof of Theorem 1.3.* We only need to prove that  $B \leq A^{-1}$  ensures that

$$A^{-r-1}(A^{2r+1}\natural_{\frac{\alpha r}{1-t+r}}(A\sharp_{1-t}B))A^{-r-1} \leq (A\sharp_{-\alpha r}B)^{-1}. \quad (2.7)$$

Notice that  $B \leq A^{-1} \iff A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq A^{-2}$ . Let  $p = 1-t$  and apply  $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq A^{-2}$  to Tanahashi inequality (Case 1), we have

$$A^{-2r} \geq (A^{-r}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{-r})^{\frac{r}{1-t+r}}. \quad (2.8)$$

Because  $-r \in [0, 1]$  and  $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq A^{-2}$ , then

$$(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r \geq A^{-2r} \quad (2.9)$$

holds by Löwner-Heinz inequality.

Continuing applying Löwner-Heinz inequality for  $\alpha \in [0, 1]$  to (2.8) and (2.9), we have

$$(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha r} \geq (A^{-r}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{-r})^{\frac{\alpha r}{1-t+r}}. \quad (2.10)$$

Notice that

$$\begin{aligned} & A^{-r-1}(A^{2r+1}\natural_{\frac{\alpha r}{1-t+r}}(A\sharp_{1-t}B))A^{-r-1} \\ &= A^{-r-1}A^{r+\frac{1}{2}}(A^{-r-\frac{1}{2}}A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{\frac{1}{2}}A^{-r-\frac{1}{2}})^{\frac{\alpha r}{1-t+r}}A^{r+\frac{1}{2}}A^{-r-1} \\ &= A^{-\frac{1}{2}}(A^{-r}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{-r})^{\frac{\alpha r}{1-t+r}}A^{-\frac{1}{2}} \end{aligned} \quad (2.11)$$

and

$$\begin{aligned}
 & (A\sharp_{-\alpha r}B)^{-1} \\
 &= (A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-\alpha r}A^{\frac{1}{2}})^{-1} \\
 &= A^{-\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha r}A^{-\frac{1}{2}}.
 \end{aligned} \tag{2.12}$$

Together with (2.10), (2.11) and (2.12), (2.7) holds obviously.  $\square$

### 3. Some corollaries of main results

In this section, we show some corollaries of main results.

By computing,

$$\begin{aligned}
 & \prod_{i=1}^n \lambda_i \left( \left( A^{-r-1} \left( A^{2r+1} \sharp_{\frac{(1-2t+r)\alpha}{1-t+r}} (A\sharp_{1-t}B) \right) A^{-r-1} \right) \left( A\sharp_{-(1-2t+r)\alpha}B \right) \right) \\
 &= \det \left( \left( A^{-r-1} \left( A^{2r+1} \sharp_{\frac{(1-2t+r)\alpha}{1-t+r}} (A\sharp_{1-t}B) \right) A^{-r-1} \right) \left( A\sharp_{-(1-2t+r)\alpha}B \right) \right) \\
 &= \det \left( A^{-r-1} \left( A^{2r+1} \sharp_{\frac{(1-2t+r)\alpha}{1-t+r}} (A\sharp_{1-t}B) \right) A^{-r-1} \right) \cdot \det \left( A\sharp_{-(1-2t+r)\alpha}B \right) \\
 &= \det A^{-r-1} \cdot \det \left( A^{2r+1} \sharp_{\frac{(1-2t+r)\alpha}{1-t+r}} (A\sharp_{1-t}B) \right) \cdot \det A^{-r-1} \cdot \det \left( A\sharp_{-(1-2t+r)\alpha}B \right) \\
 &= \det A^{-2r-2} \cdot \det A^{(2r+1)(1-\frac{(1-2t+r)\alpha}{1-t+r})} \cdot \det (A\sharp_{1-t}B)^{\frac{(1-2t+r)\alpha}{1-t+r}} \cdot \det \left( A\sharp_{-(1-2t+r)\alpha}B \right) \\
 &= \det A^{-2r-2+(2r+1)(1-\frac{(1-2t+r)\alpha}{1-t+r})} \cdot \det A^{\frac{(1-2t+r)\alpha r}{1-t+r}} \cdot \det B^{\frac{(1-2t+r)\alpha(1-t)}{1-t+r}} \\
 &\quad \cdot \det A^{1+(1-2t+r)\alpha} \cdot \det B^{-(1-2t+r)\alpha} \\
 &= \det A^{-\frac{1-2t+r}{1-t+r} \cdot r\alpha} \cdot \det B^{-\frac{1-2t+r}{1-t+r} \cdot r\alpha} \\
 &= \det (AB)^{-\frac{1-2t+r}{1-t+r} \cdot r\alpha} = \prod_{i=1}^n \lambda_i (AB)^{-\frac{1-2t+r}{1-t+r} \cdot r\alpha},
 \end{aligned}$$

we have the following corollary.

**COROLLARY 3.1.** *If  $A, B > 0$ , then*

$$\begin{aligned}
 (A^{\frac{1}{2}}BA^{\frac{1}{2}})^l & \underset{(\log)}{>} \left( A\sharp_{-(1-2t+r)\alpha}B \right)^{\frac{1}{2}} \left( A^{-r-1} \left( A^{2r+1} \sharp_{\frac{(1-2t+r)\alpha}{1-t+r}} (A\sharp_{1-t}B) \right) A^{-r-1} \right) \\
 & \quad \times \left( A\sharp_{-(1-2t+r)\alpha}B \right)^{\frac{1}{2}}
 \end{aligned}$$

holds for  $t \in [0, \frac{1}{2}]$ ,  $\alpha \in [0, 1]$  and  $1 \geq -r > 1-t \geq \frac{1}{2}$ , where  $l = -\frac{1-2t+r}{1-t+r} \cdot r\alpha$ .

Similarly, we can obtain the following corollary.

COROLLARY 3.2. *If  $A, B > 0$ , then*

$$(A^{\frac{1}{2}}BA^{\frac{1}{2}})_{(\log)}^l \succ \left(A\sharp_{-\alpha r}B\right)^{\frac{1}{2}} \left(A^{-r-1}(A^{2r+1}\sharp_{\frac{\alpha r}{1-t+r}}(A\sharp_{1-t}B))A^{-r-1}\right) \left(A\sharp_{-\alpha r}B\right)^{\frac{1}{2}}$$

*holds for  $t \in [\frac{1}{2}, 1]$ ,  $\alpha \in [0, 1]$  and  $1 \geq -r > 1-t \geq 0$ , where  $l = -\frac{\alpha r^2}{1-t+r}$ .*

Next, we show some simple corollaries directly from main results.

COROLLARY 3.3. *If  $A, B > 0$ , then*

$$\prod_{i=1}^k \lambda_i(AB)^{-r} \geq \prod_{i=1}^k \lambda_i \left( \left( A^{-r-1}(A\sharp_{1-t}B)A^{-r-1} \right) \left( A\sharp_{-(1-t+r)}B \right) \right)$$

*holds for  $t \in [0, \frac{1}{2}]$  and  $1 \geq -r > 1-t \geq \frac{1}{2}$ , where  $k = 1, 2, \dots, n$ .*

*Proof.* Notice that  $\frac{1-t+r}{1-2t+r} \in [0, 1]$ . Put  $\alpha = \frac{1-t+r}{1-2t+r}$  in Theorem 1.2.  $\square$

COROLLARY 3.4. *If  $A, B > 0$ , then*

$$\prod_{i=1}^k \lambda_i(AB)^{-r} \geq \prod_{i=1}^k \lambda_i \left( \left( A^{-r-1}(A\sharp_{1-t}B)A^{-r-1} \right) \left( A\sharp_{-(1-t+r)}B \right) \right)$$

*holds for  $t \in [\frac{1}{2}, 1]$  and  $1 \geq -r > 1-t \geq 0$ , where  $k = 1, 2, \dots, n$ .*

*Proof.* Notice that  $\frac{1-t+r}{r} \in [0, 1]$ . Put  $\alpha = \frac{1-t+r}{r}$  in Theorem 1.3.  $\square$

REMARK 3.1. If we put  $r = -1$  in Corollary 3.3 and Corollary 3.4, they are just Theorem 1.1.

REMARK 3.2. Together with Corollary 3.3 and Corollary 3.4, it is obvious that

$$\prod_{i=1}^k \lambda_i(AB)^{-r} \geq \prod_{i=1}^k \lambda_i \left( \left( A^{-r-1}(A\sharp_{1-t}B)A^{-r-1} \right) \left( A\sharp_{-(1-t+r)}B \right) \right)$$

*holds for  $1 \geq -r > 1-t \geq 0$ , where  $A, B > 0$ ,  $k = 1, 2, \dots, n$ .*

COROLLARY 3.5. *If  $A, B > 0$ , then*

$$\prod_{i=1}^k \lambda_i(AB)^{2\alpha} \geq \prod_{i=1}^k \lambda_i \left( \left( A^{-1}\sharp_{2\alpha}(A\sharp_{1-t}B) \right) \left( A\sharp_{2t\alpha}B \right) \right)$$

*holds for  $t \in [0, \frac{1}{2}]$  and  $\alpha \in [0, 1]$ , where  $k = 1, 2, \dots, n$ .*

*Proof.* Put  $r = -1$  in Theorem 1.2.  $\square$

COROLLARY 3.6. *If  $A, B > 0$ , then*

$$\prod_{i=1}^k \lambda_i(AB)^{\frac{\alpha}{t}} \geq \prod_{i=1}^k \lambda_i \left( \left( A^{-1} \sharp_{\frac{\alpha}{t}} (A \sharp_{1-t} B) \right) \left( A \sharp_{\alpha} B \right) \right)$$

*holds for  $t \in [\frac{1}{2}, 1]$  and  $\alpha \in [0, 1]$ , where  $k = 1, 2, \dots, n$ .*

*Proof.* Put  $r = -1$  in Theorem 1.3.  $\square$

REMARK 3.3. If we put  $\alpha = 1/2$  in Corollary 3.5 and  $\alpha = t$  in Corollary 3.6, they are just Theorem 1.1.

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