Operators and Matrices
Volume 13, Number 1 (2019), 301–310 doi:10.7153/oam-2019-13-20

NONLINEAR ANTI–COMMUTING MAPS OF STRICTLY TRIANGULAR MATRIX LIE ALGEBRAS

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(*Communicated by C.-K. Li*)

Abstract. Let $N(\mathbb{F})$ be the Lie algebra consisting of all strictly upper triangular $(n+1)\times(n+1)$ matrices over a field $\mathbb{F}$. A map φ on $N(\mathbb{F})$ is called to be anti-commuting if $[\varphi(x),y] = -[x,\varphi(y)]$ for any $x,y \in N(\mathbb{F})$. We show that for $n \geqslant 4$, a nonlinear map $\varphi : N(\mathbb{F}) \to N(\mathbb{F})$ is anti-commuting if and only if there exist $b,b_1,b_2 \in \mathbb{F}$ and a nonlinear function $f : N(\mathbb{F}) \to \mathbb{F}$ such that $\varphi = ad\ (bE_{2n}) + \mu_{b_2}^{(n,n+1)} + \mu_{b_1}^{(12)} + \varphi_f$, where $ad\ (bE_{2n})$ is an inner anti-commuting map, $\mu_{b_2}^{(n,n+1)}, \mu_{b_1}^{(12)}$ are extremal anti-commuting maps, φ_f is a central anti-commuting map.

1. Introduction

Let $\mathscr{A}$ be an associative ring. A map $\varphi : \mathscr{A} \to \mathscr{A}$ is called *commuting* if

$$\varphi(x)x = x\varphi(x) \ \text{ for all } x \in \mathscr{A}. \tag{1.1}$$

Let us denote the commutator or the Lie product of the elements $x,y \in \mathscr{A}$ by $[x,y] = xy - yx$. Accordingly (1.1) will be written as $[\varphi(x),x] = 0$. The identity mapping and zero mapping are two classical examples of commuting maps. The principal task when treating a commuting map is to describe its form. Linear commuting maps are closely related to biderivations. Usually we consider commuting maps imposed with some restrictions, such as additive commuting maps, commuting traces, commuting automorphisms, commuting derivations, et al. See [1, 2, 5, 6, 7, 8, 13, 14, 16, 17, 18, 23, 28]. We encourage the reader to read the well-written survey paper [4], in which the author presented the development of the theory of commuting mappings and their applications in details. Similarly, commuting maps on Lie algebras are defined. Let $\mathfrak{g}$ be a Lie algebra with Lie product $[-,-]$ over a field $\mathbb{F}$. A map $\varphi : \mathfrak{g} \to \mathfrak{g}$ is said to be *commuting* if $[\varphi(x),x] = 0$ for all $x \in \mathfrak{g}$. If the characteristic is not equal to 2, a linear map $\varphi : \mathfrak{g} \to \mathfrak{g}$ is commuting if and only if $[\varphi(x),y] = [x,\varphi(y)]$ for all $x,y \in \mathfrak{g}$. In [25], the authors determined the biderivations of parabolic subalgebras of finite dimensional simple Lie algebras. So their linear commuting maps are scalar multiplication maps. In [10,23], the authors proved that any biderivation of an infinite dimensional Schrödinger-Virasoro Lie algebra or a simple generalized Witt algebra is inner, and so

Mathematics subject classification (2010): 15A04, 15A27, 15A86.
Keywords and phrases: Strictly upper triangular matrices, anti-commuting maps.

Paper OaM-13-20

their linear commuting maps are completely determined. In [12], the authors determined the commuting automorphisms and commuting derivations of certain nilpotent Lie algebras over commutative rings. In particular, the commuting automorphisms and commuting derivations of nilradicals of finite dimensional complex simple Lie algebras are completely determined.

In recent years, more and more mathematicians are interested in discussing the nonlinear maps that preserving some property concerning Lie product (for, e.g., [9, 11, 15, 19, 20, 21, 22, 26]). In this paper, we define a nonlinear map similar to a commuting map on a Lie algebra $\mathfrak{g}$. A (may be nonlinear) map $\varphi:\mathfrak{g}\to\mathfrak{g}$ on a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is said to be *anti-commuting* if $[\varphi(x),y]=-[x,\varphi(y)]$ for all $x,y\in\mathfrak{g}$. In fact, if φ is linear, φ is a special product zero derivation of $\mathfrak{g}$ defined in [27]. If the characteristic of $\mathbb{F}$ is equal to 2, an anti-commuting map is also a commuting map. In [3], the author determined the linear commuting maps over the ring of strictly upper triangular matrices. In this paper we will determine the nonlinear anti-commuting maps on the Lie algebra consisting of all strictly upper triangular matrices. Let $\mathbb{F}$ be an arbitrary field with the characteristic char $(\mathbb{F})\neq 2$. Let $N(\mathbb{F})$ be the linear space of the $(n+1)\times(n+1)$ strictly upper triangular matrices. Denote by E the identity matrix in $N(\mathbb{F})$ and by E_{ij} the matrix with sole non-zero element 1 in the (i,j) position. Then $\{E_{ij}|1\leqslant i<j\leqslant n+1\}$ is the canonical basis of $N(\mathbb{F})$.

Set

$$N_k=\{X\in N(\mathbb{F})|X=\sum_{j-i\geqslant k}x_{ij}E_{ij}\},k=1,2,\cdots,n.$$

Then N_k are Lie ideals of the $\mathbb{F}$-algebra $N(\mathbb{F})$, $1\leqslant k\leqslant n$. Let

$$Z(N(\mathbb{F}))=\{x\in N(\mathbb{F})|[x,y]=0 \text{ for any } y\in N(\mathbb{F})\}$$

be the center of $N(\mathbb{F})$. Then $Z(N(\mathbb{F}))=N_n=\mathbb{F}E_{1,n+1}$. It is easy to see that $[N_k,N_l]\subseteq N_{k+l}$.

2. Certain standard nonlinear anti-commuting maps

We denote a nonlinear anti-commuting map by *AC*. It is easy to see that a sum of *AC*s is still a *AC*. In this section, we construct certain *AC*s, which will be used to describe nonlinear anti-commuting maps.

LEMMA 2.1. *Let* $b\in\mathbb{F}$, $X=\sum_{1\leqslant i<j\leqslant n+1}x_{ij}E_{ij}\in N(\mathbb{F})$.

(1) The inner derivation ad $(bE_{2n}):N(\mathbb{F})\to N(\mathbb{F})$ *defined by*

$$(ad\ (bE_{2n}))(X)=[bE_{2n},X]$$

for any $X\in N(\mathbb{F})$ *is a linear AC.*

(2) The map $\mu_b^{(12)}:N(\mathbb{F})\to N(\mathbb{F})$ *defined by* $\mu_b^{(12)}(X)=bx_{12}E_{2,n+1}$ *is a linear AC.*

(3) The map $\mu_b^{(n,n+1)}:N(\mathbb{F})\to N(\mathbb{F})$ *defined by* $\mu_b^{(n,n+1)}(X)=bx_{n,n+1}E_{1n}$ *is a linear AC.*

Proof. (1) It is easy to see that $ad\ (bE_{2n})$ is linear. For any $X,Y \in N(\mathbb{F})$, $[(ad\ (bE_{2n}))(X),Y]+[X,(ad\ (bE_{2n}))(Y)] = [[bE_{2n},X],Y]+[X,[bE_{2n},Y] = [bE_{2n},[X,Y]]$. Since $[X,Y] = XY - YX \in N_2$, then we may assume that

$$XY - YX = \sum_{j-i\geqslant 2} a_{ij}E_{ij}, a_{ij} \in \mathbb{F}.$$

Then $[bE_{2n},[X,Y]] = \sum_{j-i\geqslant 2} a_{ij}[bE_{2n},E_{ij}] = 0$. Thus $[(ad\ (bE_{2n}))(X),Y]+$ $[X,(ad\ (bE_{2n}))(Y)] = 0$. So $ad\ (bE_{2n})$ is a *AC*.

(2) It is easy to see that $\mu_b^{(12)}$ is linear. Let

$$X = \sum_{1\leqslant i<j\leqslant n+1} x_{ij}E_{ij}, Y = \sum_{1\leqslant i<j\leqslant n+1} y_{ij}E_{ij}.$$

Then $[\mu_b^{(12)}(X),Y] = [bx_{12}E_{2,n+1},Y] = -bx_{12}y_{12}E_{1,n+1}$, $[X,\mu_b^{(12)}(Y)] = [X,by_{12}E_{2,n+1}]$ $= bx_{12}y_{12}E_{1,n+1}$, so $[\mu_b^{(12)}(X),Y] = -[X,\mu_b^{(12)}(Y)]$.

(3) The proof is similar to that in (2). □

Next we name certain standard *AC*s.

(A) Inner *AC*.

For $b \in \mathbb{F}$, the map $ad\ (bE_{2n}) : N(\mathbb{F}) \to N(\mathbb{F})$ defined in Lemma 2.1(1) is called an inner *AC*.

(B) Extremal *AC*.

For $b \in \mathbb{F}$, $\mu_b^{(12)}$, $\mu_b^{(n,n+1)}$ defined in Lemma 2.1(2)(3) are called extremal *AC*s.

(C) Central *AC*.

Let $f : N(\mathbb{F}) \to \mathbb{F}$ be a nonlinear function. We define a nonlinear map $\varphi_f : N(\mathbb{F}) \to N(\mathbb{F})$ by $\varphi_f(X) = f(X)E_{1,n+1}$. Since $Z(N(\mathbb{F})) = \mathbb{F}E_{1,n+1}$, it is easy to see that φ_f is anti-commuting. We call φ_f a central *AC*. Note that φ_f may be nonlinear.

3. Some lemmas about anti-commuting maps on $N(\mathbb{F})$

Let φ be a nonlinear anti-commuting map on $N(\mathbb{F})$.

Assume that

$$\varphi(E_{i,i+1}) \equiv \sum_{j=1}^{n} a_{ji}E_{j,j+1} \bmod N_2 \ \text{ for } \ 1 \leqslant i \leqslant n.$$

Then φ determines a matrix

$$A(\varphi) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

LEMMA 3.1. *Let φ be a AC, $n \geqslant 3$. If $i, j, r \in \{1, 2, \cdots, n\}$, $|i-j| > 1$ and $|r-i| = 1$, then $a_{rj} = 0$.*

Proof. Consider the following equality

$$[\varphi(E_{i,i+1}), E_{j,j+1}] = -[E_{i,i+1}, \varphi(E_{j,j+1})]. \tag{3.1}$$

The coefficient of $E_{i,i+2}$ on the left-hand side of the equality (3.1) is 0. Since $|r-i| = 1$, then $r = i+1$ or $i-1$. If $r = i+1$ (respectively, $r = i-1$), the coefficient of $E_{i-1,i+1}$ on the right-hand side of the equality (3.1) is $-a_{rj}$ (respectively, a_{rj}). Thus $a_{rj} = 0$. $\square$

LEMMA 3.2. *Let φ be a AC, $n \geqslant 3$. If $i, s \in \{1, 2, \cdots, n\}$, $i \neq s$, then $a_{si} = 0$.*

Proof. We prove it in the following cases.

Case 1 $|i-s| = 1$.

Since $n \geqslant 3$, we can choose $t \in \{1, 2, \cdots, n\}$ such that $t \neq i$ or s, and $|t-i| = 1$ or $|t-s| = 1$.

Case 1.1 $|t-s| = 1$.

Assume that $s = 1$. Then $|t-s| = 1$ implies that $t = 2$, and $|i-s| = 1$ also implies that $i = 2$. So $t = i$, a contradiction. Thus $s > 1$, and so $t = s-1, i = s+1$ or $t = s+1$, $i = s-1$.

Case 1.1.1 $t = s-1, i = s+1$.

Consider the following equality

$$[\varphi(E_{s+1,s+2}), E_{s-1,s}] = -[E_{s+1,s+2}, \varphi(E_{s-1,s})]. \tag{3.2}$$

The coefficient of $E_{s-1,s+1}$ on the left side of the equality (3.2) is $-a_{s,s+1}$, and the coefficient of $E_{s-1,s+1}$ on the hand side of the equality (3.2) is 0. Thus $a_{s,s+1} = 0$, i.e., $a_{si} = 0$.

Case 1.1.2 $t = s+1, i = s-1$.

Consider the following equality

$$[\varphi(E_{s-1,s}), E_{s+1,s+2}] = -[E_{s-1,s}, \varphi(E_{s+1,s+2})]. \tag{3.3}$$

The coefficient of $E_{s,s+2}$ on the left side of the equality (3.3) is $a_{s,s-1}$, and the coefficient of $E_{s,s+2}$ on the hand side of the equality (3.3) is 0. Thus $a_{s,s-1} = 0$, i.e., $a_{si} = 0$.

Case 2 $|i-s| > 1$.

Case 2.1 $|i-s| > 2$.

If $s > 1$, then $|(s-1)-i| > 1$ and $|s-(s-1)| = 1$, then $a_{si} = 0$ by Lemma 3.1. If $s = 1$, then $i > 3$. Comparing the coefficients of E_{13} on the both sides of the following equality

$$[\varphi(E_{23}), E_{i,i+1}] = -[E_{23}, \varphi(E_{i,i+1})],$$

we have $a_{1i} = 0$, i.e., $a_{si} = 0$.

Case 2.2 $|i-s| = 2$.

In this case, $s=i+2$ or $i-2$. Assume that $s=i+2$. Comparing the coefficients of $E_{i+1,i+3}$ on the both sides of the following equality

$$[\varphi(E_{i,i+1}),E_{i+1,i+2}]=-[E_{i,i+1},\varphi(E_{i+1,i+2})],$$

we have $a_{i+2,i}=0$, i.e., $a_{si}=0$. Similarly, for $s=i-2$, comparing the coefficients of $E_{i-2,i}$ on the both sides of the following equality

$$[\varphi(E_{i-1,i}),E_{i,i+1}]=-[E_{i-1,i},\varphi(E_{i,i+1})],$$

we have $a_{i-2,i}=0$, i.e., $a_{si}=0$. $\square$

In the following, we always assume that $n\geqslant 4$.

LEMMA 3.3. *Let φ be a AC. Then $A(\varphi)=0$.*

Proof. For any $i=1,2,\cdots,n-1$, comparing the coefficients of $E_{i,i+2}$ on the both sides of the equality $[\varphi(E_{i,i+1}),E_{i+1,i+2}]=-[E_{i,i+1},\varphi(E_{i+1,i+2})]$, we have $a_{ii}=-a_{i+1,i+1}$. So $a_{11}=-a_{22}=a_{33}=\cdots=(-1)^{n-1}a_{nn}$. In particular,

$$a_{11}=-a_{44}.$$

Comparing the coefficients of E_{14} on the both sides of the equality $[\varphi(E_{12}),E_{24}]=-[E_{12},\varphi(E_{24})]$, we can see that the coefficient of E_{24} in $\varphi(E_{24})$ is $-a_{11}$. On the other hand, comparing the coefficients of E_{25} on the both sides of the equality $[\varphi(E_{24}),E_{45}]=-[E_{24},\varphi(E_{45})]$, we can see that the coefficient of E_{24} in $\varphi(E_{24})$ is $-a_{44}$. Thus

$$a_{11}=a_{44},$$

and so $a_{11}=0$ by char $(\mathbb{F})\neq 2$. Thus $a_{ii}=0$ for any $i\in\{1,2,\cdots,n\}$. By Lemma 3.2, for any $i\neq s$, $a_{si}=0$. Therefore, $A(\varphi)=0$. $\square$

LEMMA 3.4. *If φ is a AC, $\varphi(E_{i,i+1})\in N_2$, $1\leqslant i\leqslant n$, then $\varphi(E_{i,i+1})\in N_{n-1}$ for any $i\in\{1,2,\cdots,n\}$.*

Proof. We will prove that $\varphi(E_{i,i+1})\in N_k$ for $1\leqslant i\leqslant n$, $2\leqslant k\leqslant n-1$. We prove it by induction on k. By conditions, it holds for $k=2$. Assume that $\varphi(E_{i,i+1})\in N_k$ for $1\leqslant i\leqslant n$, $2\leqslant k\leqslant n-2$. Set

$$\varphi(E_{i,i+1})\equiv\sum_{j=1}^{n-k+1}a_{ji}^{(k)}E_{j,j+k} \bmod N_{k+1}, 1\leqslant i\leqslant n.$$

We will prove that $a_{ji}^{(k)}=0$ for any $j\in\{1,2,\cdots,n-k+1\}$ in the following cases.

Case 1 $j+k\leqslant n$.

Case 1.1 $j\neq i$ or $i-k$.

Consider the equality

$$[\varphi(E_{i,i+1}),E_{j+k,j+k+1}]=-[E_{i,i+1},\varphi(E_{j+k,j+k+1})]. \tag{3.4}$$

Then the coefficient of $E_{j,j+k+1}$ on the left-hand side of the equality (3.4) is $a_{ji}^{(k)}$. Since $j \neq i$ or $i-k$, then the coefficient of $E_{j,j+k+1}$ on the right-hand side of the equality (3.4) is 0. So $a_{ji}^{(k)} = 0$.

Case 1.2 $j = i = 1$.

Consider the equality

$$[\varphi(E_{12}), E_{k+1,k+2}] = -[E_{12}, \varphi(E_{k+1,k+2})]. \tag{3.5}$$

By computations, the coefficient of $E_{1,k+2}$ on the left-hand (respectively, right-hand) side of the equality (3.5) is $a_{11}^{(k)}$ (respectively, $-a_{2,k+1}^{(k)}$). So $a_{11}^{(k)} = -a_{2,k+1}^{(k)}$. By the conditions $2 \neq 1+k$ or $(k+1)-k$ and $2+k \leqslant n$, then $a_{2,k+1}^{(k)} = 0$ by Case 1.1. So $a_{11}^{(k)} = 0$, i.e., $a_{ji}^{(k)} = 0$.

Case 1.3 $j = i \geqslant 2$.

Consider the equality

$$[\varphi(E_{i,i+1}), E_{i-1,i}] = -[E_{i,i+1}, \varphi(E_{i-1,i})]. \tag{3.6}$$

By computations, the coefficient of $E_{i-1,i+k}$ on the left-hand (respectively, right-hand) side of the equality (3.6) is $-a_{ii}^{(k)}$ (respectively, 0). Then $a_{ii}^{(k)} = 0$, i.e., $a_{ji}^{(k)} = 0$.

Case 1.4 $j = i-k = 1$.

In this case, $i = k+1 \in \{3, 4, \cdots, n-1\}$. Consider the following equality

$$[\varphi(E_{i,i+1}), E_{i,i+2}] = -[E_{i,i+1}, \varphi(E_{i,i+2})]. \tag{3.7}$$

Comparing the coefficient of $E_{1,i+2}$ on the both sides of the equality (3.7), we know that $a_{1i}^{(k)} = 0$, i.e., $a_{ji}^{(k)} = 0$.

Case 1.5 $j = i-k \geqslant 2$.

Consider the equality

$$[\varphi(E_{i-k-1,i-k}), E_{i,i+1}] = -[E_{i-k-1,i-k}, \varphi(E_{i,i+1})]. \tag{3.8}$$

By computations, the coefficient of $E_{i-k-1,i}$ on the left-hand (respectively, right-hand) side of the equality (3.8) is 0 (respectively, $-a_{i-k,i}^{(k)}$). Then $a_{i-k,i}^{(k)} = 0$, i.e., $a_{ji}^{(k)} = 0$.

Case 2 $j+k = n+1$.

In this case, $j = n-k+1 \geqslant 3$. We will prove that $a_{ji}^{(k)} = a_{n-k+1,i}^{(k)} = 0$.

Case 2.1 $i \neq n-k$ or n.

By the equality

$$[\varphi(E_{i,i+1}), E_{n-k,n-k+1}] = -[E_{i,i+1}, \varphi(E_{n-k,n-k+1})], \tag{3.9}$$

we have the coefficient of $E_{n-k,n+1}$ on the left-hand (respectively, right-hand) side of the equality (3.9) is $a_{n-k+1,i}^{(k)}$ (respectively, 0), then $a_{n-k+1,i}^{(k)} = 0$.

Case 2.2 $i = n-k$.

In this case, $i \geqslant 2$. Consider the equality

$$[\varphi(E_{i,i+1}), E_{i-1,i+1}] = -[E_{i,i+1}, \varphi(E_{i-1,i+1})]. \tag{3.10}$$

Comparing the coefficient of $E_{i-1,n+1}$ on the both sides of the equality (3.10), we know that $a_{i+1,i}^{(k)}=0$, i.e., $a_{n-k+1,i}^{(k)}=0$.

Case 2.3 $i=n$.

By the equality

$$[\varphi(E_{n,n+1}),E_{n-k,n-k+1}]=-[E_{n,n+1},\varphi(E_{n-k,n-k+1})], \tag{3.11}$$

we have the coefficient of $E_{n-k,n+1}$ on the left-hand (respectively, right-hand) side of the equality (3.11) is $-a_{n-k+1,n}^{(k)}$ (respectively, $a_{n-k,n-k}^{(k)}$), then $a_{n-k+1,n}^{(k)}=-a_{n-k,n-k}^{(k)}$. By Case 1.3, $a_{n-k,n-k}^{(k)}=0$, where $n-k\geqslant 2$. So $a_{n-k+1,n}^{(k)}=0$, i.e., $a_{n-k+1,i}^{(k)}=0$. □

LEMMA 3.5. *Let* φ *be a AC. If* $\varphi(E_{i,i+1})\in N_n$, $1\leqslant i\leqslant n$, *then* $\varphi(E_{i,i+k})\in N_n$ *for any* $2\leqslant k\leqslant n-1$, $1\leqslant i\leqslant n+1-k$.

Proof. Fix a $k\in\{2,3,\cdots,n-1\}$ and $i\in\{1,2,\cdots,n+1-k\}$. Set

$$\varphi(E_{i,i+k})=\sum_{1\leqslant p<q\leqslant n+1} b_{pq}E_{pq},$$

where $b_{pq}\in\mathbb{F}$ for any $1\leqslant p<q\leqslant n+1$. At first we will prove that if $(p,q)\neq(1,n+1)$, $b_{pq}=0$. We prove it in the following cases.

Case 1 $q\leqslant n$.

By conditions,

$$[\varphi(E_{i,i+k}),E_{q,q+1}]=-[E_{i,i+k},\varphi(E_{q,q+1})]=0. \tag{3.12}$$

The coefficient of $E_{p,q+1}$ of the left-hand side of the equality (3.12) is b_{pq}, then $b_{pq}=0$.

Case 2 $q=n+1$.

In this case, $p\neq 1$ or $n+1$. By conditions,

$$[\varphi(E_{i,i+k}),E_{p-1,p}]=-[E_{i,i+k},\varphi(E_{p-1,p})]=0. \tag{3.13}$$

The coefficient of $E_{p-1,n+1}$ of the left-hand side of the equality (3.13) is $-b_{p,n+1}$, then $b_{p,n+1}=0$, i.e., $b_{pq}=0$.

Thus $\varphi(E_{i,i+k})\equiv 0 \bmod N_n$, i.e., $\varphi(E_{i,i+k})\in N_n$ for any $2\leqslant k\leqslant n-1$ and $1\leqslant i\leqslant n+1-k$. □

LEMMA 3.6. *Let* φ *be a AC. If* $\varphi(E_{i,i+1})\in N_{n-1}$, $1\leqslant i\leqslant n$, *then there exist* $b,b_1,b_2\in\mathbb{F}$ *and a nonlinear map* $f:N(\mathbb{F})\to\mathbb{F}$ *with such that*

$$\varphi=ad\ (bE_{2n})+\mu_{b_2}^{(n,n+1)}+\mu_{b_1}^{(12)}+\varphi_f.$$

Proof. Set

$$\varphi(E_{i,i+1})=a_{1i}^{(n-1)}E_{1n}+a_{2i}^{(n-1)}E_{2,n+1}+a_{1i}^{(n)}E_{1,n+1}, 1\leqslant i\leqslant n.$$

Assume that $2 \leqslant i \leqslant n-1$. By the equality

$$[\varphi(E_{i,i+1}), E_{n,n+1}] = -[E_{i,i+1}, \varphi(E_{n,n+1})]. \tag{2.20}$$

we have the coefficient of $E_{1,n+1}$ on the left-hand (respectively, right-hand) side of the equality (2.20) is $a_{1i}^{(n-1)}$ (respectively, 0), then $a_{1i}^{(n-1)} = 0$. By the equality

$$[\varphi(E_{i,i+1}), E_{12}] = -[E_{i,i+1}, \varphi(E_{12})], \tag{2.21}$$

we have the coefficient of $E_{1,n+1}$ on the left-hand (respectively, right-hand) side of the equality (2.21) is $-a_{2i}^{(n-1)}$ (respectively, 0), then $a_{2i}^{(n-1)} = 0$.

Assume that $i = 1$. By the equality

$$[\varphi(E_{12}), E_{n,n+1}] = -[E_{12}, \varphi(E_{n,n+1})],$$

we have the coefficient of $E_{1,n+1}$ on the left-hand (respectively, right-hand) side of the above equality is $a_{11}^{(n-1)}$ (respectively, $-a_{2n}^{(n-1)}$), then $a_{11}^{(n-1)} = -a_{2n}^{(n-1)}$. So

$$\varphi(E_{12}) = a_{11}^{(n-1)} E_{1n} + a_{21}^{(n-1)} E_{2,n+1} + a_{11}^{(n)} E_{1,n+1},$$

$$\varphi(E_{i,i+1}) = a_{1i}^{(n)} E_{1,n+1}, 2 \leqslant i \leqslant n-1,$$

$$\varphi(E_{n,n+1}) = a_{1n}^{(n-1)} E_{1n} - a_{11}^{(n-1)} E_{2,n+1} + a_{1n}^{(n)} E_{1,n+1}.$$

Set

$$b = -a_{11}^{(n-1)}.$$

By computations, $(ad\,(-bE_{2n}) + \varphi)(E_{12})) = a_{21}^{(n-1)} E_{2,n+1} + a_{11}^{(n)} E_{1,n+1}$, and $(ad\,(-bE_{2n}) + \varphi)(E_{n,n+1})) = a_{1n}^{(n-1)} E_{1n} + a_{1n}^{(n)} E_{1,n+1}$, $(ad\,(-bE_{2n}) + \varphi)(E_{i,i+1})) = a_{1i}^{(n)} E_{1,n+1}$, $2 \leqslant i \leqslant n-1$. Set

$$b_1 = a_{21}^{(n-1)}, b_2 = a_{1n}^{(n-1)}.$$

Then $(\mu_{-b_1}^{(12)} + \mu_{-b_2}^{(n,n+1)} + ad\,(-bE_{2n}) + \varphi)(E_{i,i+1}) = a_{1i}^{(n)} E_{1,n+1}$, $1 \leqslant i \leqslant n$. By Lemma 3.5, for any $1 \leqslant i < j \leqslant n+1$, we have $(\mu_{-b_1}^{(12)} + \mu_{-b_2}^{(n,n+1)} + ad\,(-bE_{2n}) + \varphi)(E_{ij}) \in N_n$, and so for any $X \in N(\mathbb{F})$, $[(\mu_{-b_1}^{(12)} + \mu_{-b_2}^{(n,n+1)} + ad\,(-bE_{2n}) + \varphi)(X), E_{ij}] = -[X, (\mu_{-b_1}^{(12)} + \mu_{-b_2}^{(n,n+1)} + ad\,(-bE_{2n}) + \varphi)(E_{ij})] = 0$. Thus

$$(\mu_{-b_1}^{(12)} + \mu_{-b_2}^{(n,n+1)} + ad\,(-bE_{2n}) + \varphi)(X) \in Z(N(\mathbb{F})),$$

and so there exists an element $c_X \in \mathbb{F}$ such that $(\mu_{-b_1}^{(12)} + \mu_{-b_2}^{(n,n+1)} + ad\,(-bE_{2n}) + \varphi)(X) = c_X E_{1,n+1}$. So we can define a nonlinear map $f : N(\mathbb{F}) \to \mathbb{F}$ by $f(X) = c_X$ for any $X \in N(\mathbb{F})$. Therefore $\mu_{-b_1}^{(12)} + \mu_{-b_2}^{(n,n+1)} + ad\,(-bE_{2n}) + \varphi = \varphi_f$ by definitions, and so $\varphi = -\mu_{-b_1}^{(12)} - \mu_{-b_2}^{(n,n+1)} + ad\,bE_{2n} + \varphi_f = \mu_{b_1}^{(12)} + \mu_{b_2}^{(n,n+1)} + ad\,bE_{2n} + \varphi_f$. $\square$

4. The main theorem about anti-commuting maps

THEOREM 4.1. *Let φ be a nonlinear map on $N(\mathbb{F})$, which is the Lie algebra consisting of all strictly upper $(n+1)\times(n+1)$ matrices, $n \geqslant 4$. Then φ is anti-commuting if and only if there exist $b, b_1, b_2 \in \mathbb{F}$ and a nonlinear function $f : N(\mathbb{F}) \to \mathbb{F}$ such that $\varphi = \mu_{b_1}^{(12)} + \mu_{b_2}^{(n,n+1)} + ad\ (bE_{2n}) + \varphi_f$.*

Proof. The sufficiency of the theorem is obvious.

Assume that $\varphi(E_{i,i+1}) \equiv \sum_{j=1}^{n} a_{ji}E_{j,j+1} \bmod N_2,\ 1 \leqslant i \leqslant n$, and $A(\varphi) = (a_{ij})_{n\times n}$. Then by Lemma 3.3, $A(\varphi) = 0$, and so $\varphi(E_{i,i+1}) \in N_2,\ 1 \leqslant i \leqslant n$. By Lemma 3.4, $\varphi(E_{i,i+1}) \in N_{n-1},\ 1 \leqslant i \leqslant n$. By Lemma 3.6, the necessity of the theorem holds. □

COROLLARY 4.2. *Let φ be a linear map on $N(\mathbb{F})$, $n \geqslant 4$. Then φ is anti-commuting if and only if there exist $b, b_1, b_2 \in \mathbb{F}$ and a linear function $f : N(\mathbb{F}) \to \mathbb{F}$ such that $\varphi = \mu_{b_1}^{(12)} + \mu_{b_2}^{(n,n+1)} + ad\ (bE_{2n}) + \varphi_f$.*

Proof. By Lemma 2.1, $\mu_{b_1}^{(12)}, \mu_{b_2}^{(n,n+1)}, ad\ (bE_{2n})$ are linear. It is easy to see that f is linear if and only if φ_f is linear. Then the corollary follows from the above main theorem. □

Acknowledgement. This work is supported by the National Natural Science Foundation of China (Grant No. 11871014, 11571360) and the Natural Science Foundation of Fujian Province (Grant No. 2016J01006) and JZ160427.

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(Received March 27, 2018)

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