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THE COMPRESSIONS OF THE WEIGHTED CONDITIONAL EXPECTATION OPERATORS

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Abstract. A C-E type Toeplitz operator T_u^E is the compression of a weighted conditional expectation operator EM_u to the Bergman space L_a^2. This study focuses on bounded C-E type Toeplitz operators. Several properties of such operators are obtained, in particular, those of finite rank are described. Also, for these linear operators the trace is computed and as some applications, several examples are obtained.

1. Introduction and preliminaries

Suppose (X,Σ,μ) be a sigma finite measure space and let $\mathscr{A}$ be a subalgebra of Σ. We denote that the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The associated $\mathscr{A}\subseteq\Sigma$, there exists an operator $E:=E^{\mathscr{A}}:L^p(\Sigma)\to L^p(\mathscr{A})$ which is called conditional expectation operator. $\mathscr{D}(E)$, the domain of E, contains the set of all non-negative measurable functions and each $f\in L^p(\Sigma)$ with $1\leqslant p\leqslant\infty$, which satisfies

$$\int_A f d\mu = \int_A E(f)d\mu, \quad A\in\mathscr{A}.$$

Recall that, $E:L^2(\Sigma)\to L^2(\mathscr{A})$ is an orthogonal projection onto $L^2(\mathscr{A})$. We now restrict our attention to the case $(\mathbb{D},\mathscr{M},A)$, where $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, $\mathscr{M}$ is the sigma-algebra of Lebesgue-measurable sets in $\mathbb{D}$ and A is normalized area measure in $\mathbb{D}$. For $1\leqslant p<\infty$, the Bergman space $L_a^p(\mathbb{D})=L_a^p(\mathscr{M})$ is a closed subspace of $L^p(\mathscr{M})$ consisting of analytic functions. Good references for the basic material on the Bergman spaces can be found in [2, 8]. Let P be the Bergman projection. For $u\in L^\infty(\mathscr{M})$, the operator T_u defined on $L_a^2(\mathbb{D})$ by $T_uf=P(uf)$ is called Toeplitz operator. Let φ_z be the analytic map of $\mathbb{D}$ onto $\mathbb{D}$ defined by

$$\varphi_z=\frac{z-w}{1-\bar{z}w}, \quad z\in\mathbb{D}.$$

For $z\in\mathbb{D}$, let $K_z\in L_a^2$ denote the Bergman reproducing kernel of L_a^2. As is well known,

$$K_z(w)=\frac{1}{(1-\bar{z}w)^2}.$$

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Let k_z denote the normalized Bergman reproducing kernel. Then $k_z = (1-|z|^2)K_z$ is also in L^2_a. We remind the reader that the Berezin transform of the function $u \in L^1$, on $\mathbb{D}$ is given by

$$B(u)(z) = \langle uk_z, k_z\rangle, \qquad z \in \mathbb{D}.$$

Making change variable, we have $B(u)(z) = \int_{\mathbb{D}}(u\circ\varphi_z)(w)|k_z\circ\varphi_z(w)|^2|\varphi_z'(w)|^2 dA(w)$. Since $|\varphi_z'(w)|^2 = \frac{(1-|z|^2)}{|1-\bar{z}w|^4}$, we obtain (see [1])

$$B(u)(z) = \int_{\mathbb{D}}(u\circ\varphi_z)(w)dA(w). \tag{1.1}$$

For $u \in L^1(\mathbb{D})$, the C-E type Toeplitz operator with symbol u is defined by

$$T_u^E f = PE(uf) = \int_{\mathbb{D}} E(uf)(w)K(z,w)dA(w),$$

for any bounded analytic function f on $\mathbb{D}$. for $u \in L^\infty$, T_u^E is bounded. Recall that for $1 \leqslant p \leqslant \infty$, $E := E^{\mathscr{A}} : L^p(\mathscr{M}) \to L^p(\mathscr{A})$ is a bounded operator.

Now, let $u \in L^1$ and $f \in H^\infty$. Then $uf \in L^1$. Thus, $uf \in \mathscr{D}(E)$ and $E(uf) \in L^1(\mathscr{A})$. Therefore, H^∞ is contained in the domain of the operator T_u^E and so T_u^E is always densely defined. Also, we note that $T_u^E = T_u$ whence $\mathscr{A} = \mathscr{M}$ and so $E = I$ where I is the identity operator.

The weighted conditional expectation operator EM_u have been defined as combination of multiplication operator and conditional expectation operator and several properties of these operators such as boundedness, compactness, and spectra are studied (see [3]). In fact, a C-E type Toeplitz operator T_u^E is the compression of a weighted conditional expectation operator EM_u to the Bergman space L^2_a. The systematic study of C-E type Toeplitz operators was recently spurred by the paper of the first author and Moradi [5].

This study focuses on bounded C-E type Toeplitz operators. In this paper, Several properties of such operators are obtained, in particular, those of finite rank are described. Also, for these linear operators the trace is computed and as some applications, several examples are obtained.

2. Proof of main results

There are many similarities as well as differences between the theory of Toeplitz operators and C-E type Toeplitz operators. Unlike the situation for classical Toeplitz operators on L^2_a, for a given $u \in L^1$, there many $v \in L^1$ for which $T_u^E = T_v^E$. Also, there many nontrivial compact operators whose symbols are the harmonic functions. Like a Toeplitz operator a C-E type Toeplitz operator is compact whenever its symbol has compact support. We indicate some properties of this functional operation E in the next result, without explicitly calculating it (see [7]).

THEOREM 2.1. *Let (X,Σ,μ) be a sigma finite measure space and let $\mathscr{A}$ be a subalgebra of Σ. Then $E : L^p(\mathscr{M}) \to L^p(\mathscr{A})$ is a linear order-preserving operator such that for all $f \in L^p(\mathscr{M})$,* *(i)* $\|Ef\| \leqslant \|f\|$ *(E is contraction);*

(ii) $E(fEg) = E(f)E(g)$ *(averaging identity); (iii)* $E(E(f)) = E(f)$ *(projection property).*

Thus, E is a contraction positive linear mapping on $L^p(\mathscr{M})$ with range $L^p(\mathscr{A})$. Also the preceding theorem implies $Ef = f$ if f is in the range of E, so E is the identity on $L^p(\mathscr{A})$. Furthermore, $E(\bar{f}) = \overline{Ef}$ and we always have

$$\int_{\mathbb{D}} (Ef)gdA = \int_{\mathbb{D}} (Ef)(Eg)dA = \int_{\mathbb{D}} f(Eg)dA.$$

Before proceeding, let us first recall the following lemmas from [4, 5].

LEMMA 2.2. *Suppose a and b are complex numbers and* $u, v \in L^1$*, then*
(i) $T^E_{au+bv} = aT^E_u + bT^E_v$.
(ii) $(T^E_u)^* = PM_{\bar{u}}E$.

LEMMA 2.3. *The following assertions hold.*

(i) Let $\mathscr{A} = \{\emptyset, \mathbb{D}\}$*, then* $Ef(z) = \int_{\mathbb{D}} f(w)dA(w)$.

(ii) For $1 < n \in \mathbb{N}$*, let* $\varphi(z) = z^n$*. For* $z \in \mathbb{D}$ *let* $\varphi^{-1}(\varphi(z)) = \{(z^n)^{\frac{1}{n}}\} = \{z_1, \ldots, z_n\}$*, where* $z_k = |z|e^{i\theta_k}$ *with* $\theta_k = (\arg z^n + 2k\pi)/n$*. Thus, for* $1 \leqslant k \leqslant n$*,* $|z_k| = |z|$ *and so* $|\varphi'(z_k)|$ *is not zero for all nonzero* $z \in \mathbb{D}$*. Let* $\mathscr{A} = \mathscr{A}(\varphi)$ *be the subalgebra of* $\mathscr{M}$ *generated by* $\{(z^n)^{-1}(U) : U \subset \mathbb{D}\}$ *where U is open, then for all nonzero* $z \in \mathbb{D}$ *we get*

$$E(f)(z) = \Big(\sum_{k=1}^{n} \frac{f(z_k)}{n^2|z|^{2(n-1)}}\Big)\Big(\frac{1}{n|z|^{2(n-1)}}\Big)^{-1}$$
$$= \frac{1}{n}\sum_{k=1}^{n} f(z_k).$$

Moreover, in case $n = 2$*,* $E(f)(z) = \frac{f(z)+f(-z)}{2}$*. Also, in these sigma-algebras* $EP = PE$ *and hence* $EL^p_a \subset L^p_a$.

PROPOSITION 2.4. *Suppose* $\mathscr{A} = \{\emptyset, \mathbb{D}\}$*, then* T^E_u *is compact for all* $u \in L^2$ *and* $\|T^E_u\| = \|P\bar{u}\|$.

Proof. Suppose $\mathscr{A} = \{\emptyset, \mathbb{D}\}$. Then we have

$$T^E_u f = P(E(uf)) = \int_{\mathbb{D}} u(w)f(w)dA(w) = (1 \otimes P\bar{u})f,$$

and hence T^E_u is a rank one operator. Thus, it is a compact operator. On the other hand, $\|T^E_u\| = \|(1 \otimes P\bar{u})\| = \|P\bar{u}\|$. □

REMARK 2.5. Let $\mathscr{A} = \langle C_i \rangle$ be the algebra generated by the countable collection of non-null disjoint Lebesgue measurable subset of $\mathbb{D}$ such that their union is $\mathbb{D}$. In this case (see [7]),

$$Ef = \sum_{i=1}^{\infty} \frac{1}{A(C_i)}\Big(\int_{C_i} fdA\Big)\chi_{C_i} \tag{2.1}$$

PROPOSITION 2.6. *If $\mathscr{A} = \langle C_1, \ldots, C_n \rangle$, then T_u^E is compact.*

Proof. Suppose $\mathscr{A} = \langle C_1, \ldots, C_n \rangle$. Using (2.1) we get that

$$Ef = \sum_{i=1}^{n} \frac{1}{A(C_i)} \Big(\int_{C_i} f dA \Big) \chi_{C_i}$$

and

$$E(uf) = \sum_{i=1}^{n} \frac{1}{A(C_i)} \Big(\int_{C_i} uf dA \Big) \chi_{C_i}.$$

Thus, we have

$$T_u^E f = PE(uf) = \sum_{i=1}^{n} \frac{1}{A(C_i)} \Big(\int_{C_i} uf dA \Big) P\chi_{C_i}.$$

Since T_u^E is finite rank operator, so is compact. □

EXAMPLE 2.7. Suppose $\mathscr{A} = \{\emptyset, \mathbb{D}\}$ and u is a nonzero analytic function on $\mathbb{D}$ such that $u(0) = 0$. By mean value property of harmonic functions, we get that

$$T_u^E f = P(E(uf)) = \int_{\mathbb{D}} u(w) f(w) dA(w) = uf(0) = 0.$$

Thus, unlike the situation for classical Toeplitz operators on L_a^2, for a given $u \in L^1$, there many $v \in L^1$ for which $T_u^E = T_v^E$.

EXAMPLE 2.8. Suppose again $\mathscr{A} = \{\emptyset, \mathbb{D}\}$. Then by Lemma 2.3 we have $EK_z = \int_{\mathbb{D}} K_z dA = 1$. Since $k_z \to 0$ weakly in L_a^2, so for all $g \in L_a^2$ we have $\langle k_z, g \rangle \to 0$ as $|z| \to 1^-$. Now let $f = g + h$ where $g \in L_a^2$ and $h \in (L_a^2)^\perp$. Then $\langle k_z, f \rangle = \langle k_z, g \rangle \to 0$, and so $k_z \to 0$ weakly in L^2. On the other hand,

$$\begin{aligned}
\langle T_u^E k_z, k_z \rangle &= \frac{1}{\|K_z\|} \langle T_u^E k_z, K_z \rangle = \frac{1}{\|K_z\|} \langle PE(uk_z), K_z \rangle \\
&= \frac{1}{\|K_z\|} \langle (uk_z), EK_z \rangle = \frac{1}{\|K_z\|} \int_{\mathbb{D}} uk_z \overline{EK_z} dA \\
&= (1 - |z|^2) \int_{\mathbb{D}} uk_z dA,
\end{aligned}$$

and hence $\langle T_u^E k_z, k_z \rangle \to 0$ as $|z| \to 1^-$.

In the following we present a useful sufficient condition for the boundedness of a C-E type Toeplitz operator on L_a^2.

THEOREM 2.9. *Let there exists a constant $C > 0$ and a positive measurable function h on $\mathbb{D}$ such that*

$$\int_{\mathbb{D}} |u(w) K(z,w)| h(w)^2 dA(w) \leqslant Ch(z)^2$$

and

$$\int_{\mathbb{D}} |u(w)K(z,w)|h(z)^2 dA(z) \leqslant Ch(w)^2.$$

Then T_u^E is bounded on L_a^2 with norm less than or equal to C.

Proof. This is a direct consequence of Cauchy-Schwarz inequality and Fubini's theorem. In fact, if f is in L_a^2 for any $z \in \mathbb{D}$,

$$T_u^{E*} f(z) = P(\bar{u}Ef)(z) = \int_{\mathbb{D}} \overline{u(w)} Ef(w)K(z,w)h(w)h(w)^{-1} dA(w)$$

and hence Cauchy-Schwarz inequality gives

$$|T_u^{E*} f(z)|^2 \leqslant \int_{\mathbb{D}} |u(w)K(z,w)|h(w)^2 dA(w) \int_{\mathbb{D}} |u(w)K(z,w)|h(w)^{-2}|Ef(w)|^2 dA(w).$$

By the first inequality of the assumption, we have

$$|T_u^{E*} f(z)|^2 \leqslant Ch(z)^2 \int_{\mathbb{D}} |u(w)K(z,w)|h(w)^{-2}|Ef(w)|^2 dA(w),$$

for all $z \in \mathbb{D}$. Apply Fubini's theorem again and the second inequality of the assumption, then

$$\begin{aligned} \int_{\mathbb{D}} |T_u^{E*} f(z)|^2 dA(z) &\leqslant C \int_{\mathbb{D}} h(w)^{-2} |Ef(w)|^2 dA(w) \in_{\mathbb{D}} |u(w)K(z,w)|h(z)^2 dA(z) \\ &\leqslant C^2 \int_{\mathbb{D}} |Ef(w)|^2 dA(w) \\ &\leqslant C^2 \int_{\mathbb{D}} |f(w)|^2 dA(w). \end{aligned}$$

Since T_u^{E*} is bounded, so T_u^E is a bounded operator with norm less than or equal to C. □

Let μ be a finite positive Borel measure on $\mathbb{D}$, we say μ is Carleson on L_a^2 if and only if $L_a^2 \subset L^2(\mathbb{D}, d\mu)$ and the inclusion mapping $i_2 : L_a^2 \to L^2(\mathbb{D}, d\mu)$ is bounded. Suppose μ is Carleson on L_a^2, then we say that μ is vanishing Carleson on L_a^2 if the inclusion mapping i_2 above is further compact.

PROPOSITION 2.10. *Let $B(E|u|^2)$ is bounded, then T_u^E is bounded.*

Proof. Let $B(E|u|^2)$ is bounded. Suppose $E|u|^2 dA = d\mu$, then we get

$$B(E|u|^2)(z) = \langle E|u|^2 k_z, k_z \rangle = \int_{\mathbb{D}} E|u|^2 |k_z|^2 dA = \int_{\mathbb{D}} |k_z|^2 d\mu.$$

Since $B(E|u|^2)$ is bounded, so $d\mu$ is a Carleson measure on L_a^2 [6]. Therefore, there exists a constant $C > 0$ such that

$$\begin{aligned}|\langle T_u^E f, g\rangle| &= |\langle PE(uf), g\rangle| = |\langle E(uf), g\rangle| = |\langle (uf), Eg\rangle| \\ &\leqslant \int_{\mathbb{D}} |u||f||Eg| dA \leqslant \Big(\int_{\mathbb{D}} |f|^2 dA\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{D}} |u|^2 E|g|^2 dA\Big)^{\frac{1}{2}} \\ &= \Big(\int_{\mathbb{D}} |f|^2 dA\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{D}} E|u|^2 |g|^2 dA\Big)^{\frac{1}{2}} = \Big(\int_{\mathbb{D}} |f|^2 dA\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{D}} |g|^2 d\mu\Big)^{\frac{1}{2}} \\ &\leqslant C\|f\|\|g\|,\end{aligned}$$

for all $f, g \in L_a^2$. $\square$

PROPOSITION 2.11. *Let u is $\mathscr{A}$-measurable and $T_{|u|}^E$ is bounded, then T_u^E is also bounded.*

Proof. Let $f, g \in L_a^2$ and let $|u|dA = d\mu$. Since $T_{|u|}^E$ is bounded, hence there exists a constant $C > 0$ such that

$$\begin{aligned}|\langle T_{|u|}^E f, g\rangle| &= |\langle PE(|u|f), g\rangle| = |\langle E(|u|f), g\rangle| = |\langle |u|Ef, Eg\rangle| \\ &= \Big|\int_{\mathbb{D}} |u| Ef\overline{Eg} dA\Big| = \Big|\int_{\mathbb{D}} Ef\overline{Eg} d\mu\Big| \leqslant C\|f\|\,\|g\|.\end{aligned}$$

Now suppose $f = g$, then we have

$$\int_{\mathbb{D}} |Ef|^2 d\mu \leqslant C\|f\|^2.$$

Therefore, for all $f, g \in L_a^2$, we get

$$\begin{aligned}|\langle T_u^E f, g\rangle| = |\langle PE(uf), g\rangle| &= \Big|\int_{\mathbb{D}} uEf\overline{Eg} dA\Big| \leqslant \int_{\mathbb{D}} |u||Ef||Eg| dA \\ &= \int_{\mathbb{D}} |Ef||Eg| d\mu \leqslant C\|f\|\|g\|.\end{aligned}$$

This completes the proof. $\square$

We consider the space $C_0(\mathbb{D})$ of all continuous complex-valued functions on $\mathbb{D}$ that can be uniformly approximated by continuous functions with compact support. Here, we show that like a Toeplitz operator a C-E type Toeplitz operator is compact whenever its symbol has compact support.

PROPOSITION 2.12. *Let $u \in L^1$ and has compact support K in $\mathbb{D}$ where $K \in \mathscr{A}$. Then T_u^E is bounded.*

Proof. We recall that $E(fg) = fE(g)$, if f is $\mathscr{A}$-measurable. Thus, since Ef is $\mathscr{A}$-measurable, so $E(Efg) = EfE(g)$. Now suppose $f, g \in L_a^2$. Since L^2 norm dominates the sup norm over any compact set K, so there is a constant $C_k > 0$ such that we have

$$
\begin{aligned}
|\langle T_u^E f, g\rangle| &= |\langle PE(uf), g\rangle| = |\langle E(uf), g\rangle| = \left|\int_K uf\overline{Eg}dA\right| \\
&\leqslant \int_K |u||f|E|g|dA \leqslant C_k\|f\| \int_K |u|E|g|dA \\
&= C_k\|f\| \int_K E(|u|E|g|)dA = C_k\|f\| \int_K E|u|E|g|dA \\
&= C_k\|f\| \int_K E(E|u||g|)dA = C_k\|f\| \int_K E|u||g|dA \\
&\leqslant C_k\|f\|\|g\| \int_K E|u|dA \leqslant C_k\|f\|\|g\| \int_K |u|dA \\
&\leqslant C_k\|f\|\ \|g\|\ \|u\|_1.
\end{aligned}
$$

This completes the proof. □

LEMMA 2.13. *Let $u \in L^\infty$ and has compact support K in $\mathbb{D}$, then T_u^E is compact.*

Proof. Let $g_n \to 0$ weakly in L_a^2, then $\{g_n\}$ converges uniformly on K to 0. Thus, $\|ug_n\| \to 0$ and so $\|E(ug_n)\| \to 0$ as $n \to \infty$. Hence, $\|T_u^E g_n\| \to 0$. □

PROPOSITION 2.14. *Suppose $u \in C_0(\mathbb{D})$ is bounded, then T_u^E is compact.*

Proof. Suppose $u \in C_0(\mathbb{D})$, then u can be uniformly approximated by continuous functions u_n on $\mathbb{D}$ with compact support. Since

$$\|T_u^E - T_{u_n}^E f\| = \|PE(u-u_n)f\| \leqslant \|(u-u_n)\|_\infty \|f\|,$$

we get $\|T_u^E - T_{u_n}^E\| \leqslant \|(u-u_n)\|_\infty$. Moreover, $\|(u-u_n)\|_\infty \to 0$ and so $\|T_u^E - T_{u_n}^E\| \to 0$ as $n \to \infty$. Thus, uniform convergence of symbols implies norm convergence of C-E type Toeplitz operators and by Lemma 2.13 a C-E type Toeplitz operator is compact if its symbol has compact support, we see that T_u^E is compact if $u \in C_0(\mathbb{D})$. □

THEOREM 2.15. *Let $B(|u|)(z) \to 0$ as $|z| \to 1^-$ and let $EP = PE$. Then T_u^E is compact.*

Proof. Let $|u|dA = d\mu$, then we get that

$$B(|u|)(z) = \int_{\mathbb{D}} |u||k_z|^2 dA = \int_{\mathbb{D}} |k_z|^2 d\mu.$$

Since $B(|u|)(z) \to 0$ as $|z| \to 1^-$, so $d\mu$ is vanishing Carleson [8]. Now let $f \in L_a^2$. Then we have

$$\begin{aligned}
\|(T_u^E)^* f\| &= \sup\{|\langle (T_u^E)^* f, g\rangle| : \|g\| = 1,\ g \in L_a^2\} \\
&= \sup\{|\langle P(\bar{u}Ef), g\rangle| : \|g\| = 1,\ g \in L_a^2\} \\
&\leqslant \sup\left\{ \int_{\mathbb{D}} |u||Ef||g| dA : \|g\| = 1,\ g \in L_a^2 \right\} \\
&= \sup\left\{ \int_{\mathbb{D}} |Ef||g| d\mu : \|g\| = 1,\ g \in L_a^2 \right\} \\
&\leqslant \|Ef\|_{L^2(\mu)} \sup\{\|g\|_{L^2(\mu)} : \|g\| = 1,\ g \in L_a^2\}.
\end{aligned}$$

Since μ is vanishing Carleson, μ is Carleson on L_a^2 in particular. Thus, there is a constant $C > 0$ such that $\|g\|_{L^2(\mu)} \leqslant C\|g\|$ for all $g \in L_a^2$, and hence $\|(T_u^E)^* f\| \leqslant C\|Ef\|_{L^2(\mu)}$. Now let $f_n \to 0$ weakly in L_a^2. The boundedness of E implies that $Ef_n \to 0$ weakly in L_a^2. The compactness of i_2 implies that $\|Ef_n\|_{L^2(\mu)} \to 0$, and so $\|(T_u^E)^* f_n\| \to 0$ as $n \to \infty$. □

EXAMPLE 2.16. Suppose, $d\mu = \chi_{Dr} dA$ where $D_r = \{z \in \mathbb{D} : |z| \leqslant r\}$. It is clear that $B(\chi_{Dr}) \to 0$ as $|z| \to 1^-$. Let $\varphi(z) = z^n$. According to Lemma 2.2, it follows that $EP = PE$. Therefore, T_u^E is compact.

THEOREM 2.17. *Let u is a nonnegative $\mathscr{A}$-measurable function on $\mathbb{D}$, then*

$$tr(T_u^E) = \int_{\mathbb{D}} u(w)\langle EK_w, K_w\rangle dA(w).$$

Proof. We note that, T_u^E is a positive operator on L_a^2 whence u is a nonnegative $\mathscr{A}$-measurable function on $\mathbb{D}$. Now fix an orthonormal basis $\{e_n\}$ of L_a^2, then by Fubini's theorem we get

$$\begin{aligned}
tr(T_u^E) &= \sum_{n=1}^{\infty} \langle T_u^E e_n, e_n\rangle = \sum_{n=1}^{\infty} \int_{\mathbb{D}} T_u^E e_n(z)\overline{e_n(z)} dA(z) \\
&= \sum_{n=1}^{\infty} \int_{\mathbb{D}} \langle T_u^E e_n, K_z\rangle \overline{e_n(z)} dA(z) = \int_{\mathbb{D}} \langle T_u^E \sum_{n=1}^{\infty} e_n \overline{e_n(z)}, K_z\rangle dA(z) \\
&= \int_{\mathbb{D}} \langle T_u^E K_z, K_z\rangle dA(z) = \int_{\mathbb{D}} \langle PE(uK_z), K_z\rangle dA(z) = \int_{\mathbb{D}} \langle uEK_z, EK_z\rangle dA(z) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} u(w)|EK_z(w)|^2 dA(w) dA(z) = \int_{\mathbb{D}} \int_{\mathbb{D}} u(w)|E\overline{K_w(z)}|^2 dA(w) dA(z) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} u(w)|\overline{EK_w(z)}|^2 dA(w) dA(z) = \int_{\mathbb{D}} \int_{\mathbb{D}} u(w)|EK_w(z)|^2 dA(z) dA(w) \\
&= \int_{\mathbb{D}} u(w) dA(w) \int_{\mathbb{D}} |EK_w(z)|^2 dA(z) = \int_{\mathbb{D}} u(w)\langle EK_w, EK_w\rangle dA(w) \\
&= \int_{\mathbb{D}} u(w)\langle EK_w, K_w\rangle dA(w).
\end{aligned}$$

This completes the proof. □

EXAMPLE 2.18. Let $\mathscr{A}$ be the subalgebras of $\mathscr{M}$ generated by $\{(z^2)^{-1}(u) : u \subset \mathbb{D}\}$ where u is open. Then by Lemma 2.3, $Ef(z) = \frac{f(z)+f(-z)}{2}$. Recall that, in this case, $EL_a^2 \subset L_a^2$. Thus

$$\begin{aligned} EK_w(w) &= \frac{K_w(w)+K_w(-w)}{2} \\ &= \frac{1}{2}\left(\frac{1}{(1-|w|^2)^2} + \frac{1}{(1+|w|^2)^2}\right) \\ &= \frac{1+|w|^4}{(1-|w|^4)^2}. \end{aligned}$$

Now let $u(w) = (1-|w|^4)^2$. Then $Eu = u$, and so u is $\mathscr{A}$-measurable. Since $\langle w^n, w^n \rangle = \frac{1}{n+1}$, we get that

$$\begin{aligned} tr(T_u^E) &= \int_{\mathbb{D}} (1-|w|^4)^2 \frac{1+|w|^4}{(1-|w|^4)^2} \\ &= \int_{\mathbb{D}} 1+|w|^4 = 1+\langle w^2, w^2\rangle = 1+\frac{1}{3} = \frac{4}{3}. \end{aligned}$$

PROPOSITION 2.19. *Suppose* $E|u|^2 \in C(\bar{\mathbb{D}})$ *and there is a* $z_0 \in \partial\mathbb{D}$ *such that* $E|u|^2(z_0) = 0$. *Then* $(T_u^E)^*$ *is not bounded below.*

Proof. Let $E|u|^2(z_0) = 0$. Making change variable, in light of (1.1) we have

$$\begin{aligned} \|(T_u^E)^* k_z\|^2 = \|P(\bar{u}Ek_z)\|^2 \leqslant \|(\bar{u}Ek_z)\|^2 &= \int_{\mathbb{D}} |uEk_z|^2 dA \\ &\leqslant \int_{\mathbb{D}} |u|^2 E|k_z|^2 dA = \int_{\mathbb{D}} E|u|^2|k_z|^2 dA \\ &= \int_{\mathbb{D}} E|u|^2 \circ \varphi_z dA. \end{aligned}$$

If $z_0 \in \partial\mathbb{D}$, then $\varphi_z(w) \to z_0$ as $z \to z_0$ for all $w \in \mathbb{D}$, and so

$$E|u|^2\varphi_z(w) \to E|u|^2(z_0) = 0.$$

By the dominated convergence theorem, we have $\int_{\mathbb{D}} E|u|^2 \circ \varphi_z dA \to 0$ and hence $(T_u^E)^*$ is not bounded below. □

EXAMPLE 2.20. Let $u(z) = \ln|z|$ and let $\varphi(z) = z^2$. Then by Lemma 2.3, we get $E|u(z)|^2 = |\ln|z||^2$. Therefore, there many $z_0 \in \partial\mathbb{D}$ such that $E|u(z)|^2(z_0) = 0$ and so $(T_u^E)^*$ is not bounded below.

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