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LYAPUNOV PROPERTY OF POSITIVE C_0–SEMIGROUPS ON NON–COMMUTATIVE L^p SPACES

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Abstract. That the growth bound of a positive C_0-semigroup on classical L_p-space coincides with the spectral bound of its generator, is a well known result in classical semigroup theory. In this paper we study this result in the non-commutative setting.

1. Introduction

Let $\{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup on a Banach space X with a generator A. Set $s(A) := \sup\{Re\lambda : \lambda \in \sigma(A)\}$ and $w(T) := \inf\{\lambda \in \mathbb{R} : \exists M \geqslant 1 \text{ such that } \|T(t)\| \leqslant Me^{\lambda t}\ \forall t \geqslant 0\}$, where $\sigma(A)$ is the spectrum of A. If $\dim X < \infty$, then the spectral bound $s(A)$ is equal to the growth bound $w(T)$ and this implies that the solution $u(t) = T(t)x_0$ of the initial value problem in X:

$$u'(t) = Au(t), \quad u(0) = x_0$$

decays exponentially to zero if $s(A) < 0$. The equality between $s(A)$ and $w(T)$ is not true, in general, for C_0-semigroups if $\dim X = \infty$. However, the spectral mapping theorem implies that $s(A) \leqslant w(T)$ in general. It is also known that the said equality (we shall call it the *Lyapunov property* of the semigroup T) is true for every holomorphic semigroup and there are examples of violation of Lyapunov property for C_0-semigroups even on Hilbert spaces (see [1, Section 5.1]).

On the other hand, the additional assumption of positivity, in situations where it makes sense, often verifies the Lyapunov property. For example, this is true for classical L^p-spaces, L-spaces, von-Neumann algebras, $C(\Omega)$ and $C_0(\Omega)$ [1, 11]. For positive C_0-semigroups on classical L^p spaces, the fact that the Lyapunov property holds, was proven first (i) for $p = 1$ by Derdinger in 1980 [4], (ii) for $p = 2$ in 1983 by Greiner-Nagel [5] (iii) for all $1 \leqslant p < \infty$, with some additional conditions, in [16] and [7] and finally (iv) for all positive C_0 semigroups on $L^p, 1 \leqslant p < \infty$ by Weis [17] in 1995.

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This article studies Lyapunov property for C_0-semigroups defined on non commutative L^p spaces. We show directly, using Datko's theorem, that $s(A) = w(T)$ for a positive C_0-semigroup defined on non-commutative $L^1(\mathcal{M},\tau)$ or $L^2(\mathcal{M},\tau)$ space, where $\mathcal{M}$ is a von-Neumann algebra with a normal, semifinite, faithful trace τ. Moreover, following Voigt [16], where a similar result is proven in the commutative setting, we prove that the equality holds for C_0-semigroups defined on non-commutative $L^p(\mathcal{M},\tau)$ spaces for $1 \leqslant p < \infty$, provided some additional conditions hold. We also show that the Lyapunov property holds for consistent families of positive C_0-semigroups defined on a special class of non-commutative L^p spaces - the Schatten classes.

2. Preliminaries

We briefly recall the definition of non-commutative L^p-spaces, referring the reader to [3, 14] for details. Let $\mathcal{M}$ be a von-Neumann algebra with a normal, semifinite, faithful trace τ. Let S_+ be the set of all positive $x \in \mathcal{M}$ such that $\tau(x) < \infty$ and S be linear span of S_+. Then $L^p(\mathcal{M},\tau)$ is the completion of S with respect to the norm $\|x\|_p = \tau(|x|^p)^{1/p}$, for $1 \leqslant p < \infty$. $L^p(\mathcal{M},\tau)$ can also be described as a space of unbounded operators x affiliated to $\mathcal{M}$ in a certain sense such that $\tau(|x|^p) < \infty$. We set $L^\infty(\mathcal{M},\tau) = \mathcal{M}$ equipped with the operator norm. The trace τ can be extended as continuous linear functional on $L^1(\mathcal{M},\tau)$ with $|\tau(x)| \leqslant \|x\|_1$.

The usual Hölder inequality extends to the non-commutative setting. Let $1 \leqslant r,\ p,\ q \leqslant \infty$ be such that $1/r = 1/p + 1/q$ and $x \in L^p(\mathcal{M},\tau)$, $y \in L^q(\mathcal{M},\tau)$, then $xy \in L^r(\mathcal{M},\tau)$ and

$$\|xy\|_r \leqslant \|x\|_p \|y\|_q. \tag{2.1}$$

In particular, if $r = 1$, that is, $1/p + 1/q = 1$, then for $x \in L^p(\mathcal{M},\tau)$, $y \in L^q(\mathcal{M},\tau)$, we have that $xy \in L^1(\mathcal{M},\tau)$ and

$$|\tau(xy)| \leqslant \|xy\|_1 \leqslant \|x\|_p \|y\|_q. \tag{2.2}$$

This defines a natural duality between $L^p(\mathcal{M},\tau)$ and $L^q(\mathcal{M},\tau)$ such that $\langle x,y\rangle = \tau(xy^*)$. Then for any $1 \leqslant p < \infty$, $1/p + 1/q = 1$, we have

$$L^p(\mathcal{M},\tau)^* = L^q(\mathcal{M},\tau). \tag{2.3}$$

Thus $L^1(\mathcal{M},\tau)$ is the predual of $\mathcal{M}$ and $L^p(\mathcal{M},\tau)$ is reflexive for $1 < p < \infty$. The space $L^2(\mathcal{M},\tau)$ is a Hilbert space with respect to the scalar product $(x,y) \hookrightarrow \langle x, y^*\rangle$. It is known that $\mathcal{M} \cap L^1(\mathcal{M},\tau)$ is dense in $L^p(\mathcal{M},\tau)$ for $1 < p < \infty$.

Throughout this article, we will assume that $\mathcal{M}$ is a von-Neumann algebra with a normal, faithful, semifinite trace τ unless otherwise stated.

Consider a C_0-semigroup $T = \{T(t)\}_{t \geqslant 0}$ with generator A. Setting $T'(t) = e^{-wt}T(t)$ for some $w \in \mathbb{R}$, it is clear that the generator $A - w$ satisfies $s(A-w) = s(A) - w$ and $\|T'(t)\| \leqslant Me^{(w(T)-w)t}$. If we can show that $s(A) - w < 0$ implies $(w(T) - w) < 0$, then $w(T) \leqslant s(A)$, which combined with the earlier observation would imply the Lyapunov property. Since w is arbitrary, it suffices to prove that $s(A) < 0$ implies $w(T) < 0$.

The following useful criterion for $w(T) < 0$ is very well known.

THEOREM 2.1. [1, Datko's theorem]*: The following are equivalent:*

(i) $w(T) < 0$,

(ii) $\int_0^\infty \|T(t)x\|_X^p \, dt < \infty$ for all $x \in X$, and some $p \in [1,\infty)$.

We note that each of the non-commutative L^p-spaces is a normal ordered Banach space [13], and if T is a semigroup of positive maps on X, then there is a simplification to the above theorem.

LEMMA 2.2. *Let $X = L^p(\mathscr{M},\tau), 1 \leqslant p < \infty$ and $T = \{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup on X. The following are equivalent,*

(i′) $w(T) < 0$,

(ii′) $\int_1^\infty \|T(t)x\|_X^p \, dt < \infty$ for all $x \in X_+$, *the positive cone of X, and for some $p \in [1,\infty)$.*

Proof. Only the implication (ii′) $\Rightarrow$ (i′) needs to be proven and for that it suffices to show that (ii′) $\Rightarrow$ (ii) of Theorem because continuity of T implies that $\|T(t)\| \leqslant M$ for all $t \in [0,1]$. Let $x \in X$ be a self adjoint element, such that $x = x_+ - x_-$ and $x_+, x_- \in X_+$. Then by triangle inequality

$$\|T_p(t)x\|_p = \|T_p(t)x_+ - T_p(t)x_-\|_p \leqslant \|T_p(t)x_+\|_p + \|T_p(t)x_-\|_p.$$

Thus by the Minkowski inequality, one has that

$$\left(\int_1^\infty \|T_p(t)x\|_p^p \, dt\right)^{1/p} \leqslant \left(\int_1^\infty \|T_p(t)x_+\|_p^p \, dt\right)^{1/p} + \left(\int_1^\infty \|T_p(t)x_-\|_p^p \, dt\right)^{1/p} < \infty.$$

Now let $x \in X$ be arbitrary. Then $x = x_1 + ix_2$, where x_1, x_2 are self adjoint elements of X. Again by using the triangle inequality, we get

$$\|T_p(t)x\|_p = \|T_p(t)x_1 + iT_p(t)x_2\|_p \leqslant \|T_p(t)x_1\|_p + \|T_p(t)x_2\|_p$$

and an identical reasoning gives the required result. □

We shall need the following technical result in the sequel.

LEMMA 2.3. *Let $1 \leqslant p < \infty$ and $T_p := \{T_p(t)\}_{t\geqslant 0}$ be a positive C_0-semigroup on $L^p(\mathscr{M},\tau)$. For $x \in L^p(\mathscr{M},\tau)_+$ and $\alpha > \max\{0, w(T_p)\}$, set*

$$G_\alpha(s,t) := \begin{cases} e^{-\alpha(t-s)}T_p(t)x & (0 \leqslant s \leqslant t) \\ 0 & (t < s). \end{cases} \tag{2.4}$$

Then

$$\int_1^\infty \|T_p(t)x\|_p^p \, dt \leqslant \left(\frac{\alpha}{1-e^{-\alpha}}\right)^p \tau\left(\int_0^\infty \left(\int_0^\infty G_\alpha(s,t) \, ds\right)^p dt\right). \tag{2.5}$$

Proof. For a fixed $t \in \mathbb{R}_+$

$$\int_0^\infty G_\alpha(s,t)\, ds = \int_0^t e^{-\alpha(t-s)} T_p(t)x\, ds = \left(\int_0^t e^{-\alpha(t-s)} ds\right) T_p(t)x$$
$$= \left(\frac{1-e^{-\alpha t}}{\alpha}\right) T_p(t)x \in L^p(\mathscr{M},\tau)_+,$$

since $T_p(t)x \in L^p(\mathscr{M},\tau)_+$ due to positivity of $T_p(t)$.

Thus $(\int_0^\infty G_\alpha(s,t)ds)^p \in L^1(\mathscr{M},\tau)_+$ for all $t \geqslant 0$, so that,

$$0 \leqslant \tau\left(\int_0^\infty G_\alpha(s,t)\, ds\right)^p < \infty.$$

Thus

$$\begin{aligned}
\tau\left(\int_0^\infty \left(\int_0^\infty G_\alpha(s,t)ds\right)^p dt\right) &= \tau\left(\int_0^\infty \left(\frac{1-e^{-\alpha t}}{\alpha}\right)^p (T_p(t)x)^p\, dt\right) \\
&\geqslant \tau\left(\int_1^\infty \left(\frac{1-e^{-\alpha t}}{\alpha}\right)^p (T_p(t)x)^p\, dt\right) \\
&\geqslant \left(\frac{1-e^{-\alpha}}{\alpha}\right)^p \tau\left(\int_1^\infty (T_p(t)x)^p\, dt\right) \\
&= \left(\frac{1-e^{-\alpha}}{\alpha}\right)^p \int_1^\infty \tau\,(T_p(t)x)^p\, dt \\
&= \left(\frac{1-e^{-\alpha}}{\alpha}\right)^p \int_1^\infty \|T_p(t)x\|_p^p\, dt.
\end{aligned}$$

Hence,

$$\int_1^\infty \|T_p(t)x\|_p^p\, dt \leqslant \left(\frac{\alpha}{1-e^{-\alpha}}\right)^p \tau\left(\int_0^\infty \left(\int_0^\infty G_\alpha(s,t)ds\right)^p dt\right). \quad \square$$

Throughout the rest of this article, we will assume that $\mathscr{M}$ is a von-Neumann algebra with a normal faithful semifinite trace τ unless otherwise stated.

3. The case when $p = 1,2$

In the next theorem, we give a direct proof of the fact that the Lyapunov property holds for all C_0-semigroups on $L^1(\mathscr{M},\tau)$. This result may also be deduced indirectly, from the facts that for such spaces, the norm is additive on the positive cone, these spaces are normal, ordered Banach spaces [14], and some spectral bounds of the generator of C_0-semigroups defined on such spaces coincide (see [1, Section 5.3]).

THEOREM 3.1. *Let $T_1 := \{T_1(t)\}_{t\geqslant 0}$ be a positive C_0-semigroup on $L^1(\mathscr{M},\tau)$, with generator A_1. Then*

$$s(A_1) = w(T_1).$$

Proof. Let $\{T_1(t)\}$ and A_1 be as above and suppose that $s(A_1) < 0$. In view of Lemma 2.2, Theorem 2.1 and the discussion preceeding it, it suffices to show that $\int_1^\infty \|T_1(t)x\|_1 dt < \infty$, for all $x \in L^1(\mathscr{M},\tau)_+$. Let $\alpha > \max\{0, w(T_1)\}$, $x \in L^1(\mathscr{M},\tau)_+$ and G_α be as in Lemma 2.3. Due to Lemma 2.2 and Lemma 2.3, it is enough to show that

$$\tau\left(\int_0^\infty \left(\int_0^\infty G_\alpha(s,t)ds\right) dt\right) < \infty \text{ for all } x \in L^1(\mathscr{M},\tau)_+. \tag{3.1}$$

Note that the positivity of $T_1(t)$ implies that $e^{-\alpha(t-s)}T_1(t)x \in L^1(\mathscr{M},\tau)_+$ for all $t,s \in \mathbb{R}_+$. Changing the order of integration in the expression on the left hand side of (3.1), we get

$$\tau\left(\int_0^\infty \left(\int_0^\infty G_\alpha(s,t)\, ds\right) dt\right) = \tau\left(\int_0^\infty \left(\int_s^\infty e^{-\alpha(t-s)} T_1(t) x\, dt\right) ds\right), \tag{3.2}$$

and on setting $t - s = u$ in the expression on RHS of 3.2, we have

$$\begin{aligned}
\tau\left(\int_0^\infty \left(\int_s^\infty e^{-\alpha(t-s)} T_1(t) x\, dt\right) ds\right) &= \tau\left(\int_0^\infty \left(\int_0^\infty e^{-\alpha u} T_1(s+u) x\, du\right) ds\right)\\
&= \tau\left(\int_0^\infty \left(\int_0^\infty e^{-\alpha u} T_1(s) T_1(u) x\, du\right) ds\right)\\
&= \tau\left(\int_0^\infty T_1(s)\left(\int_0^\infty e^{-\alpha u} T_1(u) x\, du\right) ds\right)\\
&= \tau\left(\int_0^\infty T_1(s)\,(\phi_\alpha(x))\, ds\right) = \tau\left(R(0,A_1)\phi_\alpha(x)\right),
\end{aligned}$$

where $\phi_\alpha(x) := \int_0^\infty e^{-\alpha u} T_1(u)x\, du \in L^1(\mathscr{M},\tau)_+$ since $\alpha > w(T_1)$. Therefore, for all $x \in L^1(\mathscr{M},\tau)_+$, we have

$$\tau\left(\int_0^\infty \left(\int_0^\infty G(s,t)\, ds\right) dt\right) = \tau\left(R(0,A_1)\phi_\alpha(x)\right) \leqslant \|R(0,A_1)\| \|\phi_\alpha(x)\|_1 < \infty.$$

Therefore one has $\int_1^\infty \|T_1(t)x\|_1\, dt < \infty$, which implies the same conclusion for $\int_0^\infty \|T_1(t)x\|_1\, dt < \infty$ and hence by Lemma 2.2, the result follows. □

THEOREM 3.2. *Let $T_2 := \{T_2(t)\}_{t \geqslant 0}$ be a positive C_0-semigroup on $L^2(\mathscr{M},\tau)$, which is symmetric, that is, $\langle T_2(t)x, y\rangle = \tau((T_2(t)x)y^*) = \langle x, T_2(t)y\rangle$ for all $x,y \in L^2(\mathscr{M},\tau)$ and for all $t \geqslant 0$, with generator A_2. Then*

$$s(A_2) = w(T_2).$$

Proof. Suppose $s(A_2) < 0$. Let $\alpha > \max\{0, w(T_2)\}$, $x \in L^2(\mathscr{M},\tau)_+$ and G_α be as in Lemma 2.3. It is sufficient to show in view of Lemma 2.3, that

$$\tau\left(\int_0^\infty \left(\int_0^\infty G_\alpha(s,t)ds\right)^2 dt\right) < \infty.$$

We note that $G_\alpha(.,t) : [0,1] \longrightarrow L^2(\mathscr{M},\tau)$ is continuous and hence

$$\left(\int_0^\infty G_\alpha(s,t)\,ds\right)^2 = \left(\int_0^\infty G_\alpha(s,t)\,ds\right)\left(\int_0^\infty G_\alpha(s',t)\,ds'\right) = \iint_{I_1\cup I_2} G_\alpha(s,t)G_\alpha(s',t)\,ds\,ds',$$

where $I_1 := \left\{(s,s') \in \mathbb{R}_+^2 : 0 \leqslant s \leqslant s' \leqslant t\right\}$ and $I_2 := \left\{(s,s') \in \mathbb{R}_+^2 : 0 \leqslant s' \leqslant s \leqslant t\right\}$. Also

$$\iint_{I_1\cup I_2} G_\alpha(s,t)G_\alpha(s',t)\,ds\,ds' = \iint_{I_1\cup I_2} e^{-\alpha(2t-s-s')}(T_2(t)x)^2\,ds\,ds'.$$

Now using symmetry in (s,s'), we get

$$\begin{aligned}
\iint_{I_1\cup I_2} e^{-\alpha(2t-s-s')}(T_2(t)x)^2 dsds' &= 2\iint_{I_2} e^{-\alpha(2t-s-s')}(T_2(t)x)^2\,ds\,ds' \\
&= 2\left(\int_0^t \left(\int_0^s e^{-\alpha(2t-s-s')}(T_2(t)x)^2\,ds'\right) ds\right) \\
&= 2\left(\int_0^t e^{-\alpha(2t-s)}(T_2(t)x)^2 \left(\int_0^s e^{\alpha s'}\,ds'\right) ds\right) \\
&= \frac{2}{\alpha}\left(\int_0^t e^{-\alpha(2t-s)}(T_2(t)x)^2\,(e^{\alpha s}-1)\,ds\right) \\
&\leqslant \frac{2}{\alpha}\left(\int_0^t e^{-2\alpha(t-s)}(T_2(t)x)^2\,ds\right).
\end{aligned}$$

Thus on evaluating the trace, we get that

$$\begin{aligned}
\tau\left(\int_0^\infty \left(\int_0^\infty G(s,t)\,ds\right)^2 dt\right) &\leqslant \tau\left(\int_0^\infty \left(\frac{2}{\alpha}\left(\int_0^t e^{-2\alpha(t-s)}(T_2(t)x)^2\,ds\right)\right) dt\right) \\
&= \frac{2}{\alpha}\left(\int_0^\infty \left(\int_0^t e^{-2\alpha(t-s)}\tau(T_2(t)x)^2\,ds\right) dt\right) \\
&= \frac{2}{\alpha}\int_0^\infty \int_0^\infty \chi_{[0,t]}(s)e^{-2\alpha(t-s)}\tau(T_2(t)x)^2\,ds\,dt \\
&= \frac{2}{\alpha}\int_0^\infty \int_0^\infty \chi_{[s,\infty]}(t)e^{-2\alpha(t-s)}\tau(T_2(t)x)^2\,dt\,ds \\
&= \frac{2}{\alpha}\left(\int_0^\infty \left(\int_s^\infty e^{-2\alpha(t-s)}\tau(T_2(t)x)^2\,dt\right) ds\right) \\
&= \frac{2}{\alpha}\left(\int_0^\infty \left(\int_0^\infty e^{-2\alpha u}\tau(T_2(s+u)x)^2\,du\right) ds\right) \\
&= \frac{2}{\alpha}\left(\int_0^\infty \left(\int_0^\infty e^{-2\alpha u}\tau(T_2(s)T_2(u)x)^2\,du\right) ds\right) \\
&= \frac{2}{\alpha}\left(\int_0^\infty \left(\int_0^\infty e^{-2\alpha u}K(s,u)\,du\right) ds\right),
\end{aligned} \tag{3.3}$$

where

$$K(s,u) := \langle T_2(s)T_2(u)x, T_2(s)T_2(u)x\rangle = \langle T_2(u)x, T_2(2s)T_2(u)x\rangle,$$

due to the symmetry and semigroup property of $\{T_2(t)\}_{t\geqslant 0}$.

Hence, by a change of order of integration, which is justified by the positivity of the integrand, we have that the expression in (3.3) above

$$= \frac{2}{\alpha}\left(\int_0^\infty e^{-2\alpha u}\left\langle T_2(u)x, \left(\int_0^\infty (T_2(2s)ds\right) T_2(u)x\right\rangle du\right).$$

Since $s(A_2) < 0$, 0 is in $\rho(A_2)$ and in such a case $R(0,A_2)$ is self adjoint and one has that the above expression

$$= \frac{1}{\alpha}\left(\int_0^\infty e^{-2\alpha u}\langle T_2(u)x, R(0,A_2)T_2(u)x\rangle\, du\right) \leqslant \frac{\|R(0,A_2)\|}{\alpha}\int_0^\infty e^{-2\alpha u}\|T_2(u)x\|^2\, du,$$

which is finite since $\alpha > w(T_2)$. □

REMARK 3.3. We note that $L^2(\mathscr{M},\tau)$ is a Hilbert space which is also an ordered Banach space with normal cone [13]. Therefore, for positive C_0-semigroups on $L^2(\mathscr{M},\tau)$ the Lyapunov property holds in view of [1, Theorem 5.3.1 and Theorem 5.2.1]. In Theorem 3.2 above, we give a different and direct proof of the fact that the Lyapunov property holds for positive symmetric semigroups defined on $L^2(\mathscr{M},\tau)$.

4. Lyapunov property for consistent families of C_0- semigroups

In this section we show that the Lyapunov property holds for consistent families of positive C_0-semigroups under certain conditions.

By a **consistent family** of C_0-semigroups defined on the non-commutative L^p spaces we shall mean a family $\{T_p : 1 \leqslant p < \infty\}$ of semigroups such that for each $p, T_p := \{T_p(t)\}_{t\geqslant 0}$ is a C_0-semigroup defined on $L^p(\mathscr{M},\tau)$ and for all $t \geqslant 0, p, q \in [1,\infty)$,

$$T_p(t)x = T_q(t)x, \quad \text{for all } x \in L^p(\mathscr{M},\tau)\cap L^q(\mathscr{M},\tau). \tag{4.1}$$

REMARK 4.1. It has been shown in [3] that every C_0-semigroup $\{T_2(t)\}_{t\geqslant 0}$, defined on $L^2(\mathscr{M},\tau)$ which is symmetric and Markov (that is, $0 \leqslant T_2(t)x \leqslant 1$ for $0 \leqslant x \leqslant 1$), extends to a consistent family of C_0-semigroups on $L^p(\mathscr{M},\tau), 1 \leqslant p < \infty$.

We recall that the Schatten classes form a major example of non-commutative L^p spaces. Given a Hilbert space $H, 1 \leqslant p < \infty$, the Schatten class $S^p(H)$ is defined as

$$S^p(H) := \{A \in \mathscr{B}(H) : Tr(|A|^p) < \infty\}, \tag{4.2}$$

where $|A| := (A^*A)^{1/2}$ and Tr is the usual operator trace.

The following result is the key to proving the Lyapunov property for consistent families of C_0-semigroups on these spaces.

LEMMA 4.2. [10, Lemma 1.1] *For* $x \in S^p(H) \cap S^q(H), 1 \leqslant q \leqslant p < \infty$,

$$\|x\|_p \leqslant \|x\|_q.$$

Using Lemma 4.2 we are able to establish the following relation between spectral bounds of the generators of consistent C_0-semigroups.

THEOREM 4.3. *Let* $T := \{T_r : 1 \leqslant r < \infty\}$ *be a consistent family of positive* C_0-*semigroups on the non commutative spaces* $S^r(H)$ *and suppose that* $s(A_q) < 0$ *for some* $1 \leqslant q < \infty$. *Then* $s(A_p) < 0$ *and* $R(0,A_q)x = R(0,A_p)x$ *for all* $p \geqslant q$ *and for all* $x \in S^p(H) \cap S^q(H)$.

Proof. Let $q < p < \infty$ and $x \in S^p(H) \cap S^q(H)$. Since T represents a consistent family, therefore $T_p(t)x = T_q(t)x$. Moreover, since $s(A_q) < 0$, it follows that $R(0,A_q)$ exists as a bounded operator on $S^q(H)$ and from [1, Theorem 5.3.1] we have that

$$R(0,A_q)x = \int_0^\infty T_q(t)x\, dt, \quad \forall\, x \in S^q(H). \tag{4.3}$$

Since $\int_0^\infty T_q(t)x\, dt$ exists in $S^q(H)$, Lemma 4.2, $\int_0^\infty T_p(s)x\, ds$ exists in $S^p(H)$. Moreover,

$$\int_0^\infty T_p(s)x\, ds = \int_0^\infty T_q(s)x\, ds = R(0,A_q)x. \tag{4.4}$$

Denseness of $S^p(H) \cap S^q(H)$ in $S^p(H)$ now implies that the map $y \mapsto \int_0^\infty T_p(s)y\, ds$ exists as a bounded linear operator on $S^p(H)$ and hence coincides with $R(0,A_p)$. Thus, $s(A_p) < 0$. That the resolvents agree on $S^p(H) \cap S^q(H)$ is just the equation (4.4). □

THEOREM 4.4. *Suppose* $\{T_p : 1 \leqslant p < \infty\}$ *is a consistent family of positive* C_0-*semigroups on* $S^p(H)$. *Then* $s(A_q) = w(T_q)$ *for all* $q \in [1,\infty)$.

Proof. Fix $q \in (1,2)$. Suppose $s(A_q) < 0$. Then by Theorem 4.3, $s(A_2) < 0$. But the Lyapunov property holds for positive semigroups on Hilbert spaces which are also normal ordered Banach spaces, and hence also for $S^2(H)$ (see Remark 3.3). Thus $s(A_2) = w(T_2) < 0$. Again, using Lemma 4.2 we have that for $p : 1/p + 1/q = 1$,

$$\|T_p(t)x\| \leqslant \|T_2(t)x\| \text{ for all } t \geqslant 0$$

and for all $x \in S^2(H) \cap S^p(H)$ as $p > 2$. Hence $w(T_p) \leqslant w(T_2) < 0$. Due to duality, $\|T_q(t)\| = \|(T_q(t))^*\| = \|T_p(t)\|$ for all $t \geqslant 0$ whence $w(T_q) < 0$. Thus $s(A_q) = w(T_q)$ for all $q \in (1,2)$ and by duality for all $q \in (1,2) \cup (2,\infty)$. Combining this with Theorem 3.1 and Remark 3.3, we have our result. □

It is well known that the non-commutative L^p spaces associated with a semifinite von-Nuemann algebra form an interpolation scale both with respect to the complex and real interpolation methods [14]:

$$L^p(\mathscr{M},\, \tau) = (L^{p_0}(\mathscr{M},\, \tau), L^{p_1}(\mathscr{M},\, \tau))_\theta \text{ (with equal norms)}, \tag{4.5}$$

$$L^p(\mathscr{M},\ \tau) = (L^{p_0}(\mathscr{M},\ \tau), L^{p_1}(\mathscr{M},\ \tau))_{\theta,p} \text{ (with equivalent norms)}, \tag{4.6}$$

where $1 \leqslant p_0,\ p_1 \leqslant \infty,\ 0 < \theta < 1,\ p = (1-\theta)/p_0 + \theta/p_1$ and where $(\cdot,\cdot)_\theta, (\cdot,\cdot)_{\theta,p}$ denote respectively the complex and real interpolation methods. Non-commutative version of the Reisz Thorin interpolation Theorem [14] also holds. In the following, we use these facts to obtain some relations between spectral and growth bounds of consistent families of semigroups on the non commutative L^p spaces.

THEOREM 4.5. *Let $T := \{T_r : 1 \leqslant r < \infty\}$ be a consistent family of C_0-semigroups on $L^p(\mathscr{M},\tau), 1 \leqslant p < \infty$. Suppose that $\int_0^\infty \|T_1(t)x\|_1\, dt < \infty$,* for all $x \in L^1(\mathscr{M},\tau)$ *and also that $\int_0^\infty \|T_2(t)x\|_2^2\, dt < \infty$,* for all $x \in L^2(\mathscr{M},\tau)$. *Then for each $p \in [1,\infty)$,*

$$\int_0^\infty \|T_p(t)x\|_p^p\, dt < \infty, \quad \textit{for all } x \in L^p(\mathscr{M},\tau). \tag{4.7}$$

Equivalently,

$$w(T_i) < 0,\ i = 1,2 \textit{ implies that } w(T_p) < 0, \quad \textit{for all } p \in [1,\infty).$$

Proof. Define a map $\mathscr{T}_1 : L^1(\mathscr{M},\tau) \longrightarrow L^1(\mathbb{R}_+, L^1(\mathscr{M},\tau))$ as $x \hookrightarrow \mathscr{T}_1 x$ such that $(\mathscr{T}_1 x)(t) = T_1(t)x$. Then $\mathscr{T}_1$ is a linear map. We claim that $\mathscr{T}_1$ is a closed map. Let $x_n \longrightarrow x$ in $L^1(\mathscr{M},\tau)$ such that $\mathscr{T}_1 x_n \longrightarrow y$ for some $y \in L^1(\mathbb{R}_+, L^1(\mathscr{M},\tau))$. Therefore, $\int_0^\infty \|T_1(t)x_n - y(t)\|_1\, dt \longrightarrow 0$, which in turn implies that $T(t)x_{n_k} \longrightarrow y(t)$ almost everywhere for some subsequence (x_{n_k}) of (x_n). On the other hand, boundedness of $T_1(t)$ implies that $T_1(t)x_n \longrightarrow T(t)x$. Thus $y(t) = T(t)x$ for almost all t. Since $y \in L^1(\mathbb{R}_+, L^1(\mathscr{M},\tau))$, this implies that $\mathscr{T}_1 x \in L^1(\mathbb{R}_+, L^1(\mathscr{M},\tau))$. Therefore $\mathscr{T}_1$ is a closed map defined on $L^1(\mathscr{M},\tau)$. Now the closed graph theorem implies that $\mathscr{T}_1$ is a bounded linear map.

Similarly, $\mathscr{T}_2 : L^2(\mathscr{M},\tau) \longrightarrow L^2(\mathbb{R}_+, L^2(\mathscr{M},\tau))$ defined by $(\mathscr{T}_2 x)(t) = T_2(t)x$ is a bounded linear map.

By interpolation, we have that for $1 \leqslant p \leqslant 2$, the linear operator

$$\mathscr{T}_p : L^p(\mathscr{M},\tau) \longrightarrow L^p(\mathbb{R}_+, L^p(\mathscr{M},\tau))$$

as $x \hookrightarrow \mathscr{T}_p x$ such that $(\mathscr{T}_p x)(t) = T_p(t)x$, is bounded with $\|\mathscr{T}_p x\|_{p,p} \leqslant C_p \|x\|_p$ for all $x \in L^p(\mathscr{M},\tau)$ and for some $C_p \in \mathbb{R}_+$, where

$$\|\mathscr{T}_p x\|_{p,p}^p = \int_0^\infty \|T_p(t)x\|_p^p\, dt, \quad \text{for all } x \in L^p(\mathscr{M},\tau).$$

Hence (4.7) holds. Datko's theorem 2.1 gives the equivalent form of the statement of the theorem for $1 < p < 2$. For $2 < p < \infty$, the conclusions follow by a duality argument. □

As an immediate consequence of Theorem 4.5 we have the following result.

COROLLARY 4.6. *Let $T := \{T_r : 1 \leqslant r < \infty\}$ be a consistent family of C_0-semigroups on $L^p(\mathscr{M},\tau), 1 \leqslant p < \infty$ with A_p the generator of the semigroup $\{T_p(t)\}_{t \geqslant 0}$. If $s(A_1) < 0$ and $s(A_2) < 0$, then $s(A_p) < 0$ for all $p \in [1,\infty)$.*

Proof. Fix $p \in (1,\infty)$. From Theorem 3.1 and Remark 3.3 respectively we have that $w(T_1) < 0$ and $w(T_2) < 0$. Theorem 4.5 now gives $w(T_p) < 0$. Hence $s(A_p) \leqslant w(T_p) < 0$. $\square$

Under the additional assumption of independence of the growth bound of the C_0-semigroup $\{T_p(t)\}_{t\geqslant 0}$, of the parameter p the Lyapunov property can be shown to hold for consistent family of positive C_0-semigroups on the non-commutative L^p spaces. For the classical case this has been proven by Voigt [16], and we shall adapt that proof to our setting. Recall that for a C_0-semigroup $\{T(t)\}_{t\geqslant 0}$ with generator A the *uniform spectral bound* $s_0(A)$ is defined as

$$s_0(A) := \inf\{\alpha \in \mathbb{R} : H_\alpha \subset \rho(A) \text{ and } \sup_{\lambda \in H_\alpha} \|R(\lambda, A)\| < \infty\}, \tag{4.8}$$

where $H_\alpha := \{\lambda : Re\lambda > \alpha\}$.

For the generator A of a positive C_0-semigroup defined on an ordered Banach space with normal cone it is known that (see [1, Theorem 5.3.1]) $s(A) = s_0(A)$.

THEOREM 4.7. *Let $T_p := \{T_p(t)\}_{t\geqslant 0}$ be a C_0-semigroup on $L^p(\mathscr{M},\tau)$ with generator A_p, for $p \in [p_0,p_1]$, $p_0 < p_1$, and $p_0,p_1 \in [1,\infty)$. Assume*

$$T_p(t) = T_q(t) \quad \forall x \in L^p(\mathscr{M},\tau) \cap L^q(\mathscr{M},\tau), \tag{4.9}$$

and for all $p,q \in [p_0,p_1]$, $t \geqslant 0$.
Then for $r \in [0,1]$, if $p(r)$ is given by $1/p(r) := (1-r)/p_0 + r/p_1$, then we have

$$s_0(A_{p(r)}) \leqslant (1-r)s_0(A_{p_0}) + rs_0(A_{p_1}).$$

Proof. We assume, without loss of generality that $s_0(A_{p_0}) \leqslant s_0(A_{p_1})$. By hypothesis, for all $x \in \mathscr{Y} := L^{p_0}(\mathscr{M},\tau) \cap L^{p_1}(\mathscr{M},\tau)$ we have for sufficiently large z:

$$R(z,A_p)x = R(z,A_0)x = R(z,A_1)x.$$

Now we shall show that $s_0(A_p) \leqslant s_0(A_{p_1})$. For this it is sufficient to show that if $s_0(A_{p_1}) < \delta$ then $s_0(A_p) < \delta$. Suppose $s_0(A_{p_1}) < \delta$. Then, $s_0(A_{p_0}) \leqslant s_0(A_{p_1}) < \delta$, implies that

$$H_\delta \subset \rho(A_{p_0}) \text{ and } \sup_{\lambda \in H_\delta} \|R(\lambda, A_{p_0})\|_{p_0} < \infty, \tag{4.10}$$

$$H_\delta \subset \rho(A_{p_1}) \text{ and } \sup_{\lambda \in H_\delta} \|R(\lambda, A_{p_1})\|_{p_1} < \infty. \tag{4.11}$$

Now for $\xi \in H_\delta$, $R(\xi, A_{p_0})$ is a bounded linear map on $L^{p_0}(\mathscr{M},\tau)$ and so is $R(\xi, A_{p_1})$ on $L^{p_1}(\mathscr{M},\tau)$ and the bounded operators agree on $\mathscr{Y}$. Thus,

$$R(\xi, A_{p_0}) : \mathscr{Y} \longrightarrow L^{p_0}(\mathscr{M},\tau),$$
$$R(\xi, A_{p_1}) : \mathscr{Y} \longrightarrow L^{p_1}(\mathscr{M},\tau),$$

with $\|R(\xi,A_{p_0})\|_{p_0} \leqslant M_0$, and $\|R(\xi,A_{p_1})\|_{p_1} \leqslant M_1$. Complex interpolation now yields, for $\theta \in [0,1]$, and $1/p = (1-\theta)/p_0 + \theta/p_1$, that

$$R(\xi,A_p) : \mathscr{Y} \longrightarrow L^p(\mathscr{M},\tau)$$

and $\|R(\xi,A_p)\|_p \leqslant {M_0}^{1-\theta}{M_1}^{\theta}$. Because of denseness of $\mathscr{Y}$ in $L^p(\mathscr{M},\tau)$, we can extend $R(\xi,A_p)$ to all of $L^p(\mathscr{M},\tau)$. Thus we get that $R(\xi,A_p)$ is a bounded linear map and $\sup\limits_{\xi\in H_\delta} \|R(\xi,A_p)\|_p < \infty$, for all $\xi \in H_\delta$. Therefore $s_0(A_p) < \delta$. We also have that

$$R(z,A_{p_0})x = R(z,A_p)x = R(z,A_{p_1})x, \tag{4.12}$$

for all z with $\mathrm{Re}\, z > s_0(A_{p_1})$ and for all $x \in \mathscr{Y}$.

Now we shall show that $s_0(A_{p(r)}) \leqslant (1-r)s_0(A_{p_0}) + rs_0(A_{p_1})$. It is sufficient to show that if $\hat{r} \in (0,1)$, $\alpha_j > s_0(A_{p_j})$, $(j = 0,1)$, $\alpha_0 < \alpha_1$, then

$$s_0(A_{p(\hat{r})}) \leqslant (1-\hat{r})\alpha_0 + \hat{r}\alpha_1.$$

Define $F(z)x := (z - A_{p_0})^{-1}x$, for $x \in \mathscr{Y}$, and for $\alpha_0 \leqslant \mathrm{Re}\, z \leqslant \alpha_1$. Then F is analytic on $\alpha_0 < \mathrm{Re}\, z < \alpha_1$ and continuous on its boundary $\{z \in \mathbb{C} : \mathrm{Re}\, z = \alpha_0 \text{ or } \mathrm{Re}\, z = \alpha_1\}$. From (4.12), we have $F(z) := (z - A_{p_1})^{-1}x$ for all $x \in \mathscr{Y}$ and for $\mathrm{Re}\, z = \alpha_1$, and by definition of $s_0(T_p)$, we have that

$$\max\left(\sup_{\mathrm{Re}\, z=\alpha_0} \|F(z)\|_{p_0}, \sup_{\mathrm{Re}\, z=\alpha_1} \|F(z)\|_{p_1}\right) < \infty.$$

Let

$$M := \max\left(\sup_{\mathrm{Re}\, z=\alpha_0} \|F(z)\|_{p_0}, \sup_{\mathrm{Re}\, z=\alpha_1} \|F(z)\|_{p_1}\right).$$

In view of (4.5) and [9, Theorem 2.7] we have that

$$\|F((1-r)\alpha_0 + r\alpha_1) + iy)\|_{p(r)} \leqslant M,$$

for all $r \in [0,1]$, $y \in \mathbb{R}$. In particular, for $\hat{r} \leqslant r \leqslant 1$, we have

$$\|F((1-r)\alpha_0 + r\alpha_1) + iy)\|_{p(\hat{r})} \leqslant M.$$

Therefore,

$$\|F(z)\|_{p(\hat{r})} \leqslant M,$$

for all z with $(1-\hat{r})\alpha_0 + \hat{r}\alpha_1 \leqslant \mathrm{Re}\, z \leqslant \alpha_1$. Thus $R(z,A_{p(\hat{r})})$ can be extended as a bounded holomorphic function to the strip $\alpha_0 < \mathrm{Re}\, z < \alpha_1$. Now by using (4.12), for $\mathrm{Re}\, z > s_0(T_{p(\hat{r})})$, we have that $s_0(A_{p(\hat{r})}) \leqslant (1-\hat{r})\alpha_0 + \hat{r}\alpha_1$. Hence the result. □

COROLLARY 4.8. *Suppose that $\{T_p(t)\}_{t\geqslant 0}$ is a positive C_0-semigroup on $L^p(\mathscr{M},\tau)$ for all $p \in [p_0,p_1]$, satisfying* (4.9).

(i) *Then for all* $r \in [0,1]$, *we have*

$$s(A_{p(r)}) \leqslant (1-r)s(A_{p_0}) + rs(A_{p_1}).$$

(ii) *Assume that* $p_0 < 2 < p_1$, *and that* $w(T_p)$ *is independent of* $p \in [p_0, p_1]$. *Then for all* $p \in [p_0, p_1]$, *we have*

$$s(A_p) = w(T_P).$$

Proof.

(i) Since $\{T_p(t)\}_{t\geqslant 0}$ is a positive C_0-semigroup on $L^p(\mathcal{M},\tau)$ which is an ordered Banach space with normal cone, $s_0(A_p) = s(A_p)$. Hence by Theorem 4.7, we have

$$s(A_{p(r)}) \leqslant (1-r)s(A_{p_0}) + rs(A_{p_1}).$$

(ii) Let $w_0 := w(T_q)$ for all $q \in [p_0, p_1]$. Suppose $w_0 > s(A_p)$ for some $p \in [p_0, 2)$. Then there exists $r \in (0,1)$ such that $p(r) = 2$, where $1/p(r) := (1-r)/p + r/p_1$. Thus part (i) applied to $[p, p_1]$ implies that $s(A_2) \leqslant (1-r)s(A_p) + rs(A_{p_1})$. Hence, using Remark 3.3 we have that

$$w_0 = s(A_2) \leqslant (1-r)s(A_p) + rs(A_{p_1}) < w_0,$$

which is a contradiction. The case when $w_0 > s(A_p)$ for some $p \in (2, p_1)$ can be dealt with similarly. □

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