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ON GENERALIZED DAVIS–WIELANDT RADIUS INEQUALITIES OF SEMI–HILBERTIAN SPACE OPERATORS

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Abstract. Let A be a positive (semidefinite) operator on a complex Hilbert space $\mathscr{H}$ and let $\mathbb{A} = \begin{pmatrix} A & O \\ O & A \end{pmatrix}$. We obtain upper and lower bounds for the A-Davis-Wielandt radius of semi-Hilbertian space operators, which generalize and improve on the existing ones. Further, we derive upper bounds for the A-Davis-Wielandt radius of the sum of the product of semi-Hilbertian space operators. We also obtain upper bounds for the $\mathbb{A}$-Davis-Wielandt radius of 2×2 operator matrices. Finally, we determine the exact value for the $\mathbb{A}$-Davis-Wielandt radius of two operator matrices $\begin{pmatrix} I & X \\ O & O \end{pmatrix}$ and $\begin{pmatrix} O & X \\ O & O \end{pmatrix}$, where X is a semi-Hilbertian space operator, and I, O are the identity operator, the zero operator on $\mathscr{H}$, respectively.

1. Introduction and preliminaries

Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on a complex Hilbert space $\mathscr{H}$ with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$. The letters I and O stand for the identity operator and the zero operator on $\mathscr{H}$, respectively. For $T \in \mathscr{B}(\mathscr{H})$, we denote by $\mathscr{R}(T)$ and $\mathscr{N}(T)$ the range and the null space of T, respectively. By $\overline{\mathscr{R}(T)}$ we denote the norm closure of $\mathscr{R}(T)$. Let T^* be the adjoint of T. The cone of all positive semidefinite operators is given by:

$$\mathscr{B}(\mathscr{H})^+ = \{A \in \mathscr{B}(\mathscr{H}) : \langle Ax, x\rangle \geqslant 0, \ \forall\, x \in \mathscr{H}\}.$$

Every $A \in \mathscr{B}(\mathscr{H})^+$ induces the following positive semidefinite sesquilinear form:

$$\langle \cdot, \cdot \rangle_A : \mathscr{H} \times \mathscr{H} \longrightarrow \mathbb{C},\ (x, y) \longmapsto \langle x, y\rangle_A = \langle Ax, y\rangle,$$

and the sesquilinear form induces the seminorm, given by:

$$\|x\|_A = \sqrt{\langle x, x\rangle_A},\ \ x \in \mathscr{H}.$$

This makes $\mathscr{H}$ into a semi-Hilbertian space. It is easy to observe that $\|x\|_A = 0$ if and only if $x \in \mathscr{N}(A)$. Therefore, $\|\cdot\|_A$ is a norm on $\mathscr{H}$ if and only if A is injective. Also we observe that $(\mathscr{H}, \|\cdot\|_A)$ is complete if and only if $\mathscr{R}(A)$ is closed in $\mathscr{H}$. Let us fix the alphabet A for positive (semidefinite) operator on $\mathscr{H}$ and we also fix $\mathbb{A} = \begin{pmatrix} A & O \\ O & A \end{pmatrix}$.

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DEFINITION 1.1. ([2]) Let $T \in \mathscr{B}(\mathscr{H})$. An operator $S \in \mathscr{B}(\mathscr{H})$ is called an A-adjoint of T if the equality $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ holds, for all $x, y \in \mathscr{H}$.

Therefore, S is an A-adjoint of T if and only if S is a solution of the equation $AX = T^*A$ in $\mathscr{B}(\mathscr{H})$. For $T \in \mathscr{B}(\mathscr{H})$, the existence of an A-adjoint of T is not guaranteed. The set of all operators acting on $\mathscr{H}$ that admit A-adjoints is denoted by $\mathscr{B}_A(\mathscr{H})$. It follows from Douglas Theorem [13] that

$$\mathscr{B}_A(\mathscr{H}) = \{T \in \mathscr{B}(\mathscr{H}) : \mathscr{R}(T^*A) \subseteq \mathscr{R}(A)\}.$$

By Douglas Theorem [13], we have if $T \in \mathscr{B}_A(\mathscr{H})$ then the operator equation $AX = T^*A$ has a unique solution, denoted by $T^{\sharp_A}$, satisfying $\mathscr{R}(T^{\sharp_A}) \subseteq \overline{\mathscr{R}(A)}$. For a survey of the recent results related to Douglas Theorem, we refer to [22]. Note that $T^{\sharp_A} = A^\dagger T^* A$, where $A^\dagger$ is the Moore-Penrose inverse of A (see [3]). Also, we have $AT^{\sharp_A} = T^*A$ and $T(\mathscr{N}(A)) \subseteq \mathscr{N}(A)$ for every $T \in \mathscr{B}_A(\mathscr{H})$. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be A-bounded if there exists $c > 0$ such that $\|Tx\|_A \leqslant c\|x\|_A$, for all $x \in \mathscr{H}$. We observe that $\mathscr{B}_{A^{1/2}}(\mathscr{H})$ is the collection of all A-bounded operators, i.e.,

$$\mathscr{B}_{A^{1/2}}(\mathscr{H}) = \{T \in \mathscr{B}(\mathscr{H}) : \exists\, c > 0 \text{ such that } \|Tx\|_A \leqslant c\|x\|_A,\ \forall x \in \mathscr{H}\}.$$

It is well-known that $\mathscr{B}_A(\mathscr{H})$ and $\mathscr{B}_{A^{1/2}}(\mathscr{H})$ are two subalgebras of $\mathscr{B}(\mathscr{H})$ which are neither closed nor dense in $\mathscr{B}(\mathscr{H})$. Moreover, the following inclusions

$$\mathscr{B}_A(\mathscr{H}) \subseteq \mathscr{B}_{A^{1/2}}(\mathscr{H}) \subseteq \mathscr{B}(\mathscr{H})$$

hold with equality if A is injective and has closed range. The above inclusions may be proper (see [14]). Let us now define A-selfadjoint, A-normal and A-unitary operators.

DEFINITION 1.2. ([2]) An operator $T \in \mathscr{B}(\mathscr{H})$ is called A-selfadjoint if AT is selfadjoint, i.e., $AT = T^*A$ and it is called A-positive if $AT \geqslant 0$.

Observe that if T is A-selfadjoint then $T \in \mathscr{B}_A(\mathscr{H})$. However, in general, it does not always imply $T = T^{\sharp_A}$. An operator $T \in \mathscr{B}_A(\mathscr{H})$ satisfies $T = T^{\sharp_A}$ if and only if T is A-selfadjoint and $\mathscr{R}(T) \subseteq \overline{\mathscr{R}(A)}$.

DEFINITION 1.3. ([23]) An operator $T \in \mathscr{B}_A(\mathscr{H})$ is said to be A-normal if $TT^{\sharp_A} = T^{\sharp_A}T$.

We know that every selfadjoint operator is normal. But, an A-selfadjoint operator is not necessarily A-normal (see [4, Example 5.1]).

DEFINITION 1.4. ([2]) An operator $U \in \mathscr{B}_A(\mathscr{H})$ is said to be A-unitary if $\|Ux\|_A = \|U^{\sharp_A}x\|_A = \|x\|_A$, for all $x \in \mathscr{H}$.

It was shown in [2] that an operator $U \in \mathscr{B}_A(\mathscr{H})$ is A-unitary if and only if $U^{\sharp_A}U = (U^{\sharp_A})^{\sharp_A}U^{\sharp_A} = P_A$, where P_A denotes the orthogonal projection onto $\overline{\mathscr{R}(A)}$. We mention here that if $T \in \mathscr{B}_A(\mathscr{H})$ then $T^{\sharp_A} \in \mathscr{B}_A(\mathscr{H})$ and $(T^{\sharp_A})^{\sharp_A} = P_ATP_A$.

Let $T \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$. The A-operator seminorm and the A-minimum modulus of T are defined respectively as:

$$\|T\|_A = \sup\left\{ \frac{\|Tx\|_A}{\|x\|_A} : \ _{x\in\overline{\mathscr{R}(A)},\ x\neq 0} \right\} = \sup\{\|Tx\|_A : x \in \mathscr{H}, \|x\|_A = 1\}$$

and

$$m_A(T) = \inf\left\{ \frac{\|Tx\|_A}{\|x\|_A} : \ _{x\in\overline{\mathscr{R}(A)},\ x\neq 0} \right\} = \inf\{\|Tx\|_A : x \in \mathscr{H}, \|x\|_A = 1\}.$$

Let $T \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$. The A-numerical range, the A-numerical radius and the A-Crawford number of T are defined respectively as:

$$\begin{aligned} W_A(T) &= \{\langle Tx, x\rangle_A : x \in \mathscr{H}, \|x\|_A = 1\}, \\ w_A(T) &= \sup\{|c| : c \in W_A(T)\} \text{ and} \\ c_A(T) &= \inf\{|c| : c \in W_A(T)\}. \end{aligned}$$

The A-operator seminorm attainment set of T, denoted as M_T^A, is defined as the set of all A-unit vectors in $\mathscr{H}$ at which T attains its A-operator seminorm, i.e.,

$$M_T^A = \{x \in \mathscr{H} : \|Tx\|_A = \|T\|_A, \|x\|_A = 1\}.$$

Likewise the A-numerical radius attainment set and the A-Crawford number attainment set of T, denoted as W_T^A and c_T^A respectively, are defined as:

$$W_T^A = \{x \in \mathscr{H} : |\langle Tx, x\rangle_A| = w_A(T), \|x\|_A = 1\}$$

and

$$c_T^A = \{x \in \mathscr{H} : |\langle Tx, x\rangle_A| = c_A(T), \|x\|_A = 1\}.$$

It is well known that $\|\cdot\|_A$ and $w_A(\cdot)$ are equivalent seminorm on $\mathscr{B}_{A^{1/2}}(\mathscr{H})$, satisfying the following inequality (see [5]):

$$\tfrac{1}{2}\|T\|_A \leqslant w_A(T) \leqslant \|T\|_A, \ \ T \in \mathscr{B}_{A^{1/2}}(\mathscr{H}).$$

The first inequality becomes equality if $AT^2 = O$ and the second inequality becomes equality if T is A-normal (see [14]). Various results about the A-numerical radius of semi-Hilbertian space operators have been obtained, we refer the readers to [9, 10, 14, 15, 25, 26] and the references therein. For $T \in \mathscr{B}_A(\mathscr{H})$, we write $Re_A(T) = \frac{1}{2}(T + T^{\sharp_A})$ and $Im_A(T) = \frac{1}{2i}(T - T^{\sharp_A})$. For every A-selfadjoint operator T, we have (see [26])

$$w_A(T) = \|T\|_A.$$

Also $T^{\sharp_A}T$, $TT^{\sharp_A}$ are A-selfadjoint and A-positive operators satisfying the following equality:

$$\|T^{\sharp_A}T\|_A = \|TT^{\sharp_A}\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2.$$

For $T,S \in \mathscr{B}_A(\mathscr{H})$, $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$, $\|TS\|_A \leqslant \|T\|_A\|S\|_A$ and $\|Tx\|_A \leqslant \|T\|_A\|x\|_A$, for all $x \in \mathscr{H}$. For further readings we refer the readers to [2, 3].

Motivated by the study of the A-numerical radius of semi-Hilbertian space operators, we here study the A-Davis-Wielandt radius of semi-Hilbertian space operators. This is a generalization of the Davis-Wielandt radius of Hilbert space operators. The Davis-Wielandt shell and the Davis-Wielandt radius of an operator $T \in \mathscr{B}(\mathscr{H})$ are defined respectively as (see [12, 24]):

$$DW(T) = \left\{\left(\langle Tx,x\rangle, \|Tx\|^2\right) : x \in \mathscr{H}, \|x\| = 1\right\}$$

and

$$dw(T) = \sup\left\{\sqrt{|\langle Tx,x\rangle|^2 + \|Tx\|^4} : x \in \mathscr{H}, \|x\| = 1\right\}.$$

Recently many mathematicians [18, 19, 20, 27, 28] have studied the Davis-Wielandt shell and the Davis-Wielandt radius of an operator $T \in \mathscr{B}(\mathscr{H})$. The A-Davis-Wielandt shell and the A-Davis-Wielandt radius of an operator $T \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$ are defined respectively as (see [17]):

$$DW_A(T) = \left\{\left(\langle Tx,x\rangle_A, \|Tx\|_A^2\right) : x \in \mathscr{H}, \|x\|_A = 1\right\}$$

and

$$dw_A(T) = \sup\left\{\sqrt{|\langle Tx,x\rangle_A|^2 + \|Tx\|_A^4} : x \in \mathscr{H}, \|x\|_A = 1\right\}.$$

It is easy to see that the A-Davis-Wielandt radius of $T \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$ satisfying the following inequality:

$$\max\{w_A(T), \|T\|_A^2\} \leqslant dw_A(T) \leqslant \sqrt{w_A^2(T) + \|T\|_A^4}. \tag{1}$$

Recently, Feki in [16] have obtained some upper bounds for the A-Davis-Wielandt radius of operators in $\mathscr{B}_A(\mathscr{H})$.

In section 2, we find the equality conditions of the lower bound for the A-Davis-Wielandt radius of A-bounded operators mentioned in (1). We obtain upper and lower bounds for the A-Davis-Wielandt radius of operators in $\mathscr{B}_A(\mathscr{H})$, which generalize and improve on the existing ones. Further, we obtain inequalities for the $\mathbb{A}$-Davis-Wielandt radius of 2×2 operator matrices in $\mathscr{B}_{\mathbb{A}}(\mathscr{H}\oplus\mathscr{H})$. Next, we obtain upper bounds for the A-Davis-Wielandt radius of the sum of the product operators in $\mathscr{B}_A(\mathscr{H})$, i.e., if $P,Q,X,Y \in \mathscr{B}_A(\mathscr{H})$ then for any $t \in \mathbb{R}\setminus\{0\}$, we have

$$dw_A^2(PXQ^{\sharp_A} \pm QYP^{\sharp_A}) \leqslant (t^2\|P\|_A^2 + \frac{1}{t^2}\|Q\|_A^2)^2\{(t^2\|PX\|_A^2 + \frac{1}{t^2}\|QY\|_A^2)^2 + \alpha^2\}$$

and

$$dw_A^2(P^{\sharp_A}XQ \pm Q^{\sharp_A}YP) \leqslant (t^2\|P\|_A^2 + \frac{1}{t^2}\|Q\|_A^2)^2\{(t^2\|YP\|_A^2 + \frac{1}{t^2}\|XQ\|_A^2)^2 + \alpha^2\},$$

where $\alpha = w_{\mathbb{A}}\begin{pmatrix} O & X \\ Y & O \end{pmatrix}$. Finally, we compute the exact value for the $\mathbb{A}$-Davis-Wielandt radius of two operator matrices $\begin{pmatrix} I & X \\ O & O \end{pmatrix}$ and $\begin{pmatrix} O & X \\ O & O \end{pmatrix}$, where $X \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$.

2. Main results

We begin this section with the study of the equality conditions of both upper and lower bounds of A-bounded operators mentioned in (1). Fisrt we mention the following known result (see [17, Th. 11 and Prop. 4]).

THEOREM 2.1. *Let $T \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$. Then the following conditions are equivalent:*

(i) $dw_A(T) = \sqrt{w_A^2(T) + \|T\|_A^4}$.

(ii) *T is A-normaloid, i.e, $w_A(T) = \|T\|_A$.*

(iii) *There exist a sequence of A-unit vectors $\{x_n\}$ in $\mathscr{H}$ such that*

$$\lim_{n\to\infty} \|Tx_n\|_A = \|T\|_A \text{ and } \lim_{n\to\infty} |\langle Tx_n, x_n\rangle_A| = w_A(T).$$

REMARK 2.2. If $\mathscr{H}$ is finite-dimensional then condition (iii) of Theorem 2.1 is replaced by $M_T^A \cap W_T^A \neq \emptyset$, i.e., there exists an A-unit vector x in $\mathscr{H}$ such that $\|Tx\|_A = \|T\|_A$ and $|\langle Tx, x\rangle_A| = w_A(T)$.

Now, in the following two theorems we find the equality conditions of the first inequality in (1).

THEOREM 2.3. *Let $T \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$. Then the following conditions are equivalent:*

(i) $dw_A(T) = w_A(T)$.

(ii) $AT = O$.

Proof. The part $(ii) \Rightarrow (i)$ follows trivially. We only prove $(i) \Rightarrow (ii)$. Since $T \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$, there exists a sequence $\{x_n\}$ in $\mathscr{H}$ with $\|x_n\|_A = 1$ such that $w_A(T) = \lim_{n\to\infty} |\langle Tx_n, x_n\rangle_A|$. The sequence $\{\|Tx_n\|_A\}$, being a bounded sequence of real numbers has a convergent subsequence $\{\|Tx_{n_k}\|_A\}$. Now $w_A^2(T) = dw_A^2(T) \geqslant |\langle Tx_{n_k}, x_{n_k}\rangle_A|^2 + \|Tx_{n_k}\|_A^4$. Taking limit on both sides, we get $w_A^2(T) = dw_A^2(T) \geqslant w_A^2(T) + \lim_{k\to\infty} \|Tx_{n_k}\|_A^4$. This implies that $\lim_{k\to\infty} \|Tx_{n_k}\|_A = 0$. Therefore, it follows from Cauchy-Schwarz inequality that $w_A(T) = \lim_{k\to\infty} |\langle Tx_{n_k}, x_{n_k}\rangle_A| \leqslant \lim_{k\to\infty} \|Tx_{n_k}\|_A = 0$. So, we get $w_A(T) = 0$ and hence, $AT = O$. $\square$

THEOREM 2.4. *Let $T \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$ and $dw_A(T) = \|T\|_A^2$. Then either of the following condition holds:*

(i) *Let $M_T^A \neq \emptyset$. Then $|\langle Tx, x\rangle_A| = 0$ if $x \in M_T^A$, i.e., $M_T^A \subseteq c_T^A$.*

(ii) *Let $M_T^A = \emptyset$. Then there exists a sequence $\{x_n\}$ in $\mathscr{H}$ with $\|x_n\|_A = 1$ such that $\lim_{n\to\infty} \|Tx_n\|_A = \|T\|_A$ and $\lim_{n\to\infty} |\langle Tx_n, x_n\rangle_A| = 0$.*

Proof. (i) Let $M_T^A \neq \emptyset$ and $x \in M_T^A$. So, $\|Tx\|_A^4 = \|T\|_A^4 = dw_A^2(T) \geqslant |\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4$. This implies that $|\langle Tx, x\rangle_A| = 0$. So $x \in c_T^A$. Therefore, $M_T^A \subseteq c_T^A$.

(ii) Let $M_T^A = \emptyset$. Since $T \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$, there exists a sequence $\{x_n\}$ in $\mathscr{H}$ with $\|x_n\|_A = 1$ such that $\|T\|_A = \lim_{n\to\infty} \|Tx_n\|_A$. Since $\{|\langle Tx_n, x_n\rangle_A|\}$ is a bounded sequence of scalars, so it has a convergent subsequence $\{|\langle Tx_{n_k}, x_{n_k}\rangle_A|\}$. Now $\|T\|_A^4 = dw_A^2(T) \geqslant |\langle Tx_{n_k}, x_{n_k}\rangle_A|^2 + \|Tx_{n_k}\|_A^4$. Taking limit on both sides, we get $\|T\|_A^4 = dw_A^2(T) \geqslant \lim_{k\to\infty} |\langle Tx_{n_k}, x_{n_k}\rangle_A|^2 + \|T\|_A^4$ and so, $\lim_{k\to\infty} |\langle Tx_{n_k}, x_{n_k}\rangle_A| = 0$. This completes the proof. $\square$

REMARK 2.5. We note that the converse part of Theorem 2.4 may not hold. As for example, we consider $T = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$, $\lambda \in \mathbb{C}$ and $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then by simple calculations we have, $|\langle Tx, x\rangle_A| = 0$ for all $x \in M_T^A$, i.e., $M_T^A \subseteq c_T^A$. But, $dw_A(T) \neq \|T\|_A^2$ as $dw_A(T) \geqslant \sqrt{\frac{1}{16} + \frac{1}{64}} > \frac{1}{4} = \|T\|_A^2$.

Next we obtain lower bounds for the A-Davis-Wielandt radius of operators in $\mathscr{B}_A(\mathscr{H})$.

THEOREM 2.6. *Let $T \in \mathscr{B}_A(\mathscr{H})$. Then*

$$(i)\ dw_A^2(T) \geqslant \max\left\{ w_A^2(T) + c_A^2(T^{\sharp_A}T), \|T\|_A^4 + c_A^2(T) \right\},$$

$$(ii)\ dw_A^2(T) \geqslant 2\max\left\{ w_A(T)c_A(T^{\sharp_A}T), c_A(T)\|T\|_A^2 \right\}.$$

Proof. (i) Let x be an A-unit vector in $\mathscr{H}$. Then from the definition of $dw_A(T)$, we get

$$\begin{aligned} dw_A^2(T) &\geqslant |\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4 \\ &= |\langle Tx, x\rangle_A|^2 + \langle T^{\sharp_A}Tx, x\rangle_A^2 \\ &\geqslant |\langle Tx, x\rangle_A|^2 + c_A^2(T^{\sharp_A}T). \end{aligned}$$

Therefore, taking supremum over all A-unit vectors in $\mathscr{H}$, we have

$$dw_A^2(T) \geqslant w_A^2(T) + c_A^2(T^{\sharp_A}T).$$

Again from $dw_A^2(T) \geqslant |\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4$, where $\|x\|_A = 1$, we get

$$dw_A^2(T) \geqslant c_A^2(T) + \|Tx\|_A^4.$$

Taking supremum over all A-unit vectors in $\mathscr{H}$, we have

$$dw_A^2(T) \geqslant c_A^2(T) + \|T\|_A^4.$$

This completes the proof of (i).

(ii) For all $x \in \mathscr{H}$ with $\|x\|_A = 1$, we have

$$|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4 \geqslant 2|\langle Tx, x\rangle_A| \|Tx\|_A^2$$

and so,

$$dw_A^2(T) \geqslant 2|\langle Tx,x\rangle_A|\langle T^{\sharp_A}Tx,x\rangle_A \geqslant 2|\langle Tx,x\rangle_A|c_A(T^{\sharp_A}T).$$

Taking supremum over all A-unit vectors in $\mathscr{H}$, we get

$$dw_A^2(T) \geqslant 2w_A(T)c_A(T^{\sharp_A}T).$$

Again from $|\langle Tx,x\rangle_A|^2+\|Tx\|_A^4 \geqslant 2|\langle Tx,x\rangle_A|\|Tx\|_A^2$, we have

$$dw_A^2(T) \geqslant 2c_A(T)\|Tx\|_A^2.$$

Taking supremum over all A-unit vectors in $\mathscr{H}$, we get

$$dw_A^2(T) \geqslant 2c_A(T)\|T\|_A^2.$$

This completes the proof. □

REMARK 2.7. (i) It is easy to observe that the lower bound of the A-Davis-Wielandt radius of $T \in \mathscr{B}_A(\mathscr{H})$ obtained in Theorem 2.6 (i) is sharper than that in (1).

(ii) Also, both the inequalities in [6, Th. 2.1] follow from Theorem 2.6 by considering $A = I$.

In the following theorem we obtain an upper bound for the A-Davis-Wielandt radius of operators in $\mathscr{B}_A(\mathscr{H})$.

THEOREM 2.8. *Let $T \in \mathscr{B}_A(\mathscr{H})$. Then*

$$dw_A^2(T) \leqslant \sup_{\theta\in\mathbb{R}} w_A^2(e^{\mathrm{i}\theta}T+T^{\sharp_A}T)-2c_A(T)m_A^2(T).$$

Proof. Let $x \in \mathscr{H}$ with $\|x\|_A = 1$. Then there exists $\theta \in \mathbb{R}$ such that $|\langle Tx,x\rangle_A| = e^{\mathrm{i}\theta}\langle Tx,x\rangle_A$. Now,

$$\begin{aligned}|\langle Tx,x\rangle_A|^2+\|Tx\|_A^4 &= \langle e^{\mathrm{i}\theta}Tx,x\rangle_A^2+\langle T^{\sharp_A}Tx,x\rangle_A^2\\ &= (\langle e^{\mathrm{i}\theta}Tx,x\rangle_A+\langle T^{\sharp_A}Tx,x\rangle_A)^2-2\langle e^{\mathrm{i}\theta}Tx,x\rangle_A\langle T^{\sharp_A}Tx,x\rangle_A.\end{aligned}$$

Hence,

$$\begin{aligned}&2\langle e^{\mathrm{i}\theta}Tx,x\rangle_A\langle T^{\sharp_A}Tx,x\rangle_A+|\langle Tx,x\rangle_A|^2+\|Tx\|_A^4 = \Big(\langle e^{\mathrm{i}\theta}Tx,x\rangle_A+\langle T^{\sharp_A}Tx,x\rangle_A\Big)^2\\ \Rightarrow\ &2\langle e^{\mathrm{i}\theta}Tx,x\rangle_A\langle T^{\sharp_A}Tx,x\rangle_A+|\langle Tx,x\rangle_A|^2+\|Tx\|_A^4 = \langle(e^{\mathrm{i}\theta}T+T^{\sharp_A}T)x,x\rangle_A^2\\ \Rightarrow\ &2|\langle Tx,x\rangle_A|\langle T^{\sharp_A}Tx,x\rangle_A+|\langle Tx,x\rangle_A|^2+\|Tx\|_A^4 \leqslant w_A^2(e^{\mathrm{i}\theta}T+T^{\sharp_A}T).\end{aligned}$$

Therefore,

$$2|\langle Tx,x\rangle_A|\ \langle T^{\sharp_A}Tx,x\rangle_A+|\langle Tx,x\rangle_A|^2+\|Tx\|_A^4 \leqslant \sup_{\theta\in\mathbb{R}} w_A^2(e^{\mathrm{i}\theta}T+T^{\sharp_A}T)$$

and so,

$$2c_A(T)m_A^2(T)+|\langle Tx,x\rangle_A|^2+\|Tx\|_A^4 \leqslant \sup_{\theta\in\mathbb{R}} w_A^2(e^{\mathrm{i}\theta}T+T^{\sharp_A}T).$$

Hence, taking supremum over all A-unit vectors in $\mathscr{H}$, we get

$$2c_A(T)m_A^2(T)+dw_A^2(T) \leqslant \sup_{\theta\in\mathbb{R}} w_A^2(e^{\mathrm{i}\theta}T+T^{\sharp_A}T).$$

$$\Rightarrow dw_A^2(T) \leqslant \sup_{\theta\in\mathbb{R}} w_A^2(e^{\mathrm{i}\theta}T+T^{\sharp_A}T)-2c_A(T)m_A^2(T). \quad \square$$

Next we obtain the following upper and lower bounds for the A-Davis-Wielandt radius of operators in $\mathscr{B}_A(\mathscr{H})$.

THEOREM 2.9. *Let $T\in\mathscr{B}_A(\mathscr{H})$. Then*

$$\begin{aligned}\frac{1}{2}\left\{w_A^2(T+T^{\sharp_A}T)+c_A^2(T-T^{\sharp_A}T)\right\} &\leqslant dw_A^2(T)\\ &\leqslant \frac{1}{2}\left\{w_A^2(T+T^{\sharp_A}T)+w_A^2(T-T^{\sharp_A}T)\right\}.\end{aligned}$$

Proof. Let $x\in\mathscr{H}$ with $\|x\|_A=1$. Then

$$\begin{aligned}|\langle Tx,x\rangle_A|^2+\|Tx\|_A^4 &= \frac{1}{2}|\langle Tx,x\rangle_A+\langle Tx,Tx\rangle_A|^2+\frac{1}{2}|\langle Tx,x\rangle_A-\langle Tx,Tx\rangle_A|^2\\ &= \frac{1}{2}\left|\langle Tx,x\rangle_A+\langle T^{\sharp_A}Tx,x\rangle_A\right|^2+\frac{1}{2}\left|\langle Tx,x\rangle_A-\langle T^{\sharp_A}Tx,x\rangle_A\right|^2\\ &= \frac{1}{2}\left|\langle (T+T^{\sharp_A}T)x,x\rangle_A\right|^2+\frac{1}{2}\left|\langle (T-T^{\sharp_A}T)x,x\rangle_A\right|^2\\ &\geqslant \frac{1}{2}\left\{\left|\langle (T+T^{\sharp_A}T)x,x\rangle_A\right|^2+c_A^2(T-T^{\sharp_A}T)\right\}.\end{aligned}$$

Therefore, taking supremum over all A-unit vectors in $\mathscr{H}$, we get

$$dw_A^2(T)\geqslant \frac{1}{2}\left\{w_A^2(T+T^{\sharp_A}T)+c_A^2(T-T^{\sharp_A}T)\right\}.$$

Again,

$$\begin{aligned}|\langle Tx,x\rangle_A|^2+\|Tx\|_A^4 &= \frac{1}{2}|\langle Tx,x\rangle_A+\langle Tx,Tx\rangle_A|^2+\frac{1}{2}|\langle Tx,x\rangle_A-\langle Tx,Tx\rangle_A|^2\\ &= \frac{1}{2}\left|\langle Tx,x\rangle_A+\langle T^{\sharp_A}Tx,x\rangle_A\right|^2+\frac{1}{2}\left|\langle Tx,x\rangle_A-\langle T^{\sharp_A}Tx,x\rangle_A\right|^2\\ &= \frac{1}{2}\left|\langle (T+T^{\sharp_A}T)x,x\rangle_A\right|^2+\frac{1}{2}\left|\langle (T-T^{\sharp_A}T)x,x\rangle_A\right|^2\\ &\leqslant \frac{1}{2}\left\{w_A^2(T+T^{\sharp_A}T)+w_A^2(T-T^{\sharp_A}T)\right\}.\end{aligned}$$

Therefore, taking supremum over all A-unit vectors in $\mathscr{H}$, we get

$$dw_A^2(T)\leqslant \frac{1}{2}\left\{w_A^2(T+T^{\sharp_A}T)+w_A^2(T-T^{\sharp_A}T)\right\}. \quad \square$$

REMARK 2.10. We would like to remark that the inequality obtained in Theorem 2.9 generalizes the inequality in [6, Th. 2.2].

In the next theorem we obtain upper bounds for the A-Davis-Wielandt radius of $T \in \mathscr{B}_A(\mathscr{H})$. First we need the following lemma.

LEMMA 2.11. *Let $x, y, e \in \mathscr{H}$ with $\|e\|_A = 1$. Then*

$$|\langle x, e\rangle_A \langle e, y\rangle_A| \leqslant \frac{1}{2}\left(|\langle x, y\rangle_A| + \|x\|_A\|y\|_A\right).$$

Proof. For all $a, b, c, d \in \mathbb{R}$, we have $(ac - bd)^2 \geqslant (a^2 - b^2)(c^2 - d^2)$. Using this and the Cauchy Schwarz inequality, we get

$$\begin{aligned}
&|\langle x - \langle x, e\rangle_A e, y - \langle y, e\rangle_A e\rangle_A|^2 \leqslant \|x - \langle x, e\rangle_A e\|_A^2 \|y - \langle y, e\rangle_A e\|_A^2 \\
&\implies |\langle x, y\rangle_A - \langle x, e\rangle_A \langle e, y\rangle_A|^2 \leqslant (\|x\|_A^2 - |\langle x, e\rangle_A|^2)(\|y\|_A^2 - |\langle y, e\rangle_A|^2) \\
&\implies |\langle x, y\rangle_A - \langle x, e\rangle_A \langle e, y\rangle_A|^2 \leqslant (\|x\|_A\|y\|_A - |\langle x, e\rangle_A||\langle y, e\rangle_A|)^2.
\end{aligned}$$

Since $|\langle x, e\rangle_A| \leqslant \|x\|_A$ and $|\langle y, e\rangle_A| \leqslant \|y\|_A$, so $(\|x\|_A\|y\|_A - |\langle x, e\rangle_A||\langle y, e\rangle_A|) \geqslant 0$. Therefore,

$$\begin{aligned}
&|\langle x, y\rangle_A - \langle x, e\rangle_A \langle e, y\rangle_A| \leqslant \|x\|_A\|y\|_A - |\langle x, e\rangle_A||\langle y, e\rangle_A| \\
\implies &|\langle x, e\rangle_A \langle e, y\rangle_A| - |\langle x, y\rangle_A| \leqslant \|x\|_A\|y\|_A - |\langle x, e\rangle_A||\langle e, y\rangle_A|.
\end{aligned}$$

Hence,

$$2|\langle x, e\rangle_A \langle e, y\rangle_A| \leqslant |\langle x, y\rangle_A| + \|x\|_A\|y\|_A.$$

This completes the proof of the lemma. □

THEOREM 2.12. *Let $T \in \mathscr{B}_A(\mathscr{H})$. Then the following inequalities hold:*

$$\begin{aligned}
(i)\ \ & dw_A^2(T) \leqslant \left\| T^{\sharp_A}T + (T^{\sharp_A}T)^2 \right\|_A, \\
(ii)\ \ & dw_A^2(T) \leqslant \frac{1}{2}\left(w_A(T^2) + \|T\|_A^2\right) + \|T\|_A^4.
\end{aligned}$$

Proof. Let $x \in \mathscr{H}$ with $\|x\|_A = 1$. Then using Lemma 2.11 we get,

$$\begin{aligned}
|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4 &= |\langle Tx, x\rangle_A \langle x, Tx\rangle_A| + \langle T^{\sharp_A}Tx, x\rangle_A \langle x, T^{\sharp_A}Tx\rangle_A \\
&\leqslant \frac{1}{2}(\|Tx\|_A^2 + \langle Tx, Tx\rangle_A) + \frac{1}{2}(\|T^{\sharp_A}Tx\|_A^2 + \langle T^{\sharp_A}Tx, T^{\sharp_A}Tx\rangle_A) \\
&= \langle T^{\sharp_A}Tx, x\rangle_A + \langle (T^{\sharp_A}T)^{\sharp_A} T^{\sharp_A}Tx, x\rangle_A \\
&= \langle (T^{\sharp_A}T + (T^{\sharp_A}T)^{\sharp_A} T^{\sharp_A}T)x, x\rangle_A.
\end{aligned}$$

Now $T^{\sharp_A}T$ being an A-selfadjoint operator and $\mathscr{R}(T^{\sharp_A}T) \subseteq \overline{\mathscr{R}(A)}$, we have $(T^{\sharp_A}T)^{\sharp_A} = T^{\sharp_A}T$. Therefore,

$$|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4 \leqslant \langle (T^{\sharp_A}T + (T^{\sharp_A}T)^2 x, x\rangle_A.$$

Therefore, taking supremum over all A-unit vectors in $\mathscr{H}$, we get the inequality (i). Again considering $|\langle Tx,x\rangle_A|^2 = |\langle Tx,x\rangle_A\langle x,T^{\sharp_A}x\rangle_A|$ and then using Lemma 2.11, we get the inequality (ii). $\square$

REMARK 2.13. It is well-known that if T is A-normaloid then $\|T^2\|_A = \|T\|_A^2$. Therefore, it is easy to observe that both the inequalities in Theorem 2.12 becomes equality if T is A-normaloid.

In the next theorem we obtain an upper bound for the A-Davis-Wielandt radius of operators in $\mathscr{B}_A(\mathscr{H})$. For this we need the following lemma which follows from Lemma 2.11.

LEMMA 2.14. *Let $x,y,e \in \mathscr{H}$ with $\|e\|_A = 1$. Then*

$$\|x\|_A^2\|y\|_A^2 - |\langle x,y\rangle_A|^2 \geqslant 2|\langle x,e\rangle_A\langle e,y\rangle_A|\left(\|x\|_A\|y\|_A - |\langle x,y\rangle_A|\right).$$

THEOREM 2.15. *Let $T \in \mathscr{B}_A(\mathscr{H})$. Then*

$$\begin{aligned} dw_A^2(T) \leqslant 3&\left\|\left(T^{\sharp_A}T\right)^2 + T^{\sharp_A}T\right\|_A \\ &-c_A(T^{\sharp_A}T+T)m_A(T^{\sharp_A}T+T) - c_A(T^{\sharp_A}T-T)m_A(T^{\sharp_A}T-T). \end{aligned}$$

Proof. Let $x \in \mathscr{H}$ with $\|x\|_A = 1$. Then using Lemma 2.14 and Lemma 2.11 we get,

$$\begin{aligned} |\langle Tx,x\rangle_A|^2 &\leqslant \|Tx\|_A^2\|x\|_A^2 - 2|\langle Tx,x\rangle_A\langle x,x\rangle_A|(\|Tx\|_A\|x\|_A - |\langle Tx,x\rangle_A|) \\ &= \|Tx\|_A^2 + 2|\langle Tx,x\rangle_A||\langle x,Tx\rangle_A| - 2|\langle Tx,x\rangle_A|\|Tx\|_A \\ &\leqslant \|Tx\|_A^2 + \|Tx\|_A^2 + \langle Tx,Tx\rangle_A - 2c_A(T)\|Tx\|_A \\ &\leqslant 3\langle T^{\sharp_A}Tx,x\rangle_A - 2c_A(T)m_A(T). \end{aligned}$$

Using the above inequality, we get

$$\begin{aligned} &|\langle Tx,x\rangle_A|^2 + \|Tx\|_A^4 \\ =\ & \frac{1}{2}\left(|\|Tx\|_A^2 + \langle Tx,x\rangle_A|^2 + |\|Tx\|_A^2 - \langle Tx,x\rangle_A|^2\right) \\ =\ & \frac{1}{2}\left(|\langle (T^{\sharp_A}T+T)x,x\rangle_A|^2 + |\langle (T^{\sharp_A}T-T)x,x\rangle_A|^2\right) \\ \leqslant\ & \frac{1}{2}\Big(3\left\langle \left|T^{\sharp_A}T+T\right|_A^2 x,x\right\rangle_A - 2c_A(T^{\sharp_A}T+T)m_A(T^{\sharp_A}T+T) \\ &+3\left\langle \left|T^{\sharp_A}T-T\right|_A^2 x,x\right\rangle_A - 2c_A(T^{\sharp_A}T-T)m_A(T^{\sharp_A}T-T)\Big) \\ =\ & \frac{3}{2}\left\langle \left(\left|T^{\sharp_A}T+T\right|_A^2 + \left|T^{\sharp_A}T-T\right|_A^2\right)x,x\right\rangle_A - c_A(T^{\sharp_A}T+T)m_A(T^{\sharp_A}T+T) \\ &-c_A(T^{\sharp_A}T-T)m_A(T^{\sharp_A}T-T) \end{aligned}$$

$$\begin{aligned}
&= 3\left\langle\left((T^{\sharp_A}T)^{\sharp_A}T^{\sharp_A}T + T^{\sharp_A}T\right)x,x\right\rangle_A \\
&\quad -c_A(T^{\sharp_A}T+T)m_A(T^{\sharp_A}T+T) - c_A(T^{\sharp_A}T-T)m_A(T^{\sharp_A}T-T) \\
&= 3\left\langle\left((T^{\sharp_A}T)^2 + T^{\sharp_A}T\right)x,x\right\rangle_A \\
&\quad -c_A(T^{\sharp_A}T+T)m_A(T^{\sharp_A}T+T) - c_A(T^{\sharp_A}T-T)m_A(T^{\sharp_A}T-T).
\end{aligned}$$

Therefore, taking supremum over all A-unit vectors in $\mathscr{H}$, we get the required inequality. □

Next we prove the following lemma.

LEMMA 2.16. *Let $x,y \in \mathscr{H}$ and $\lambda \in \mathbb{C}$. Then we have the following equality:*

$$\|x\|_A^2\|y\|_A^2 - |\langle x,y\rangle_A|^2 = \|x-\lambda y\|_A^2\|y\|_A^2 - |\langle x-\lambda y,y\rangle_A|^2.$$

Proof. We have,

$$\begin{aligned}
&\|x-\lambda y\|_A^2\|y\|_A^2 - |\langle x-\lambda y,y\rangle_A|^2 \\
&= \langle x-\lambda y, x-\lambda y\rangle_A\|y\|_A^2 - |\langle x,y\rangle_A - \lambda\|y\|_A^2|^2 \\
&= \left(\|x\|_A^2 + |\lambda|^2\|y\|_A^2 - 2Re(\overline{\lambda}\langle x,y\rangle_A)\right)\|y\|_A^2 - |\langle x,y\rangle_A|^2 - |\lambda|^2\|y\|_A^4 \\
&\quad +2Re(\overline{\lambda}\langle x,y\rangle_A)\|y\|_A^2 \\
&= \|x\|_A^2\|y\|_A^2 - |\langle x,y\rangle_A|^2. \quad \square
\end{aligned}$$

Using Lemma 2.16, we obtain the following upper bound for the A-Davis-Wielandt radius of operators in $\mathscr{B}_A(\mathscr{H})$.

THEOREM 2.17. *Let $T \in \mathscr{B}_A(\mathscr{H})$. Then*

$$\begin{aligned}
dw_A^2(T) \leqslant \inf_{\lambda\in\mathbb{R}}\sup_{\theta\in\mathbb{R}}\Big\{&2|\lambda|\|\cos\theta Re_A(T) + T^{\sharp_A}T + \sin\theta Im_A(T) - \lambda I\|_A \\
&+\frac{1}{2}\|\cos\theta Re_A(T) + T^{\sharp_A}T + \sin\theta Im_A(T) - 2\lambda I\|_A^2 \\
&+\frac{1}{2}\|\cos\theta Re_A(T) - T^{\sharp_A}T + \sin\theta Im_A(T)\|_A^2\Big\}.
\end{aligned}$$

In particular,

$$\begin{aligned}
dw_A^2(T) \leqslant \frac{1}{2}\sup_{\theta\in\mathbb{R}}\Big\{&\left\|\cos\theta\ Re_A(T) + T^{\sharp_A}T + \sin\theta\ Im_A(T)\right\|_A^2 \\
&+\left\|\cos\theta\ Re_A(T) - T^{\sharp_A}T + \sin\theta\ Im_A(T)\right\|_A^2\Big\}.
\end{aligned}$$

Proof. Let $x \in \mathscr{H}$ with $\|x\|_A = 1$. Then there exists $\theta \in \mathbb{R}$ such that $|\langle Tx,x\rangle_A| = e^{-\mathrm{i}\theta}\langle Tx,x\rangle_A$. Using the Cartesian decomposition of T, i.e., $T = Re_A(T) + \mathrm{i}\, Im_A(T)$, we get,

$$\begin{aligned}
|\langle Tx,x\rangle_A| &= \langle e^{-\mathrm{i}\theta} Tx,x\rangle_A \\
&= \langle((\cos\theta - \mathrm{i}\sin\theta)(Re_A(T) + \mathrm{i}\, Im_A(T)))x,x\rangle_A \\
&= \langle(\cos\theta Re_A(T) + \sin\theta Im_A(T))x,x\rangle_A + \mathrm{i}\langle(\cos\theta Im_A(T) - \sin\theta Re_A(T))x,x\rangle_A.
\end{aligned}$$

Since $|\langle Tx,x\rangle_A| \in \mathbb{R}$, $|\langle Tx,x\rangle_A| = \langle(\cos\theta Re_A(T) + \sin\theta Im_A(T))x,x\rangle_A$. Now using Lemma 2.16, we get for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
|\langle Tx,x\rangle_A|^2 &= |\langle(\cos\theta Re_A(T) + \sin\theta Im_A(T))x,x\rangle_A|^2 \\
&= \|(\cos\theta Re_A(T) + \sin\theta Im_A(T))x\|_A^2 \\
&\quad - \|(\cos\theta Re_A(T) + \sin\theta Im_A(T))x - \lambda x\|_A^2 \\
&\quad + |\langle(\cos\theta Re_A(T) + \sin\theta Im_A(T))x - \lambda x, x\rangle_A|_A^2 \\
&= \langle(\cos\theta Re_A(T) + \sin\theta Im_A(T))^2 x,x\rangle_A \\
&\quad - \langle(\cos\theta Re_A(T) + \sin\theta Im_A(T) - \lambda I)^2 x,x\rangle_A \\
&\quad + |\langle(\cos\theta Re_A(T) + \sin\theta Im_A(T) - \lambda I)x,x\rangle_A|^2 \\
&= \Big\langle\Big\{(\cos\theta Re_A(T) + \sin\theta Im_A(T))^2 \\
&\quad - (\cos\theta Re_A(T) + \sin\theta Im_A(T) - \lambda I)^2\Big\}x,x\Big\rangle_A \\
&\quad + |\langle(\cos\theta Re_A(T) + \sin\theta Im_A(T) - \lambda I)x,x\rangle_A|^2 \\
&= \langle(2\lambda(\cos\theta Re_A(T) + \sin\theta Im_A(T)) - \lambda^2 I)x,x\rangle_A \\
&\quad + |\langle(\cos\theta Re_A(T) + \sin\theta Im_A(T) - \lambda I)x,x\rangle_A|^2.
\end{aligned}$$

Similarly, using Lemma 2.16, we have

$$\begin{aligned}
\|Tx\|_A^4 &= |\langle T^{\sharp_A} Tx,x\rangle_A|^2 \\
&= \langle(2\lambda T^{\sharp_A} T - \lambda^2 I)x,x\rangle_A + |\langle(T^{\sharp_A} T - \lambda I)x,x\rangle_A|^2.
\end{aligned}$$

Now,

$$\begin{aligned}
|\langle Tx,x\rangle_A|^2 + \|Tx\|_A^4 &= \langle 2\lambda\{\cos\theta Re_A(T) + T^{\sharp_A} T + \sin\theta Im_A(T)\}x,x\rangle_A - 2\lambda^2 \\
&\quad + \frac{1}{2}|\langle(\cos\theta Re_A(T) + T^{\sharp_A} T + \sin\theta Im_A(T) - 2\lambda I)x,x\rangle_A|^2 \\
&\quad + \frac{1}{2}|\langle(\cos\theta Re_A(T) - T^{\sharp_A} T + \sin\theta Im_A(T))x,x\rangle_A|^2 \\
&\leqslant 2|\lambda|\|\cos\theta Re_A(T) + T^{\sharp_A} T + \sin\theta Im_A(T) - \lambda I\|_A \\
&\quad + \frac{1}{2}\|\cos\theta Re_A(T) + T^{\sharp_A} T + \sin\theta Im_A(T) - 2\lambda I\|_A^2 \\
&\quad + \frac{1}{2}\|\cos\theta Re_A(T) - T^{\sharp_A} T + \sin\theta Im_A(T)\|_A^2
\end{aligned}$$

$$\leqslant \sup_{\theta\in\mathbb{R}}\Big\{2|\lambda|\|\cos\theta Re_A(T)+T^{\sharp_A}T+\sin\theta Im_A(T)-\lambda I\|_A$$
$$+\frac{1}{2}\|\cos\theta Re_A(T)+T^{\sharp_A}T+\sin\theta Im_A(T)-2\lambda I\|_A^2$$
$$+\frac{1}{2}\|\cos\theta Re_A(T)-T^{\sharp_A}T+\sin\theta Im_A(T)\|_A^2\Big\}.$$

Therefore, taking supremum over all A-unit vectors in $\mathscr{H}$, we get

$$dw_A^2(T)\leqslant \sup_{\theta\in\mathbb{R}}\Big\{2|\lambda|\|\cos\theta Re_A(T)+T^{\sharp_A}T+\sin\theta Im_A(T)-\lambda I\|_A$$
$$+\frac{1}{2}\|\cos\theta Re_A(T)+T^{\sharp_A}T+\sin\theta Im_A(T)-2\lambda I\|_A^2$$
$$+\frac{1}{2}\|\cos\theta Re_A(T)-T^{\sharp_A}T+\sin\theta Im_A(T)\|_A^2\Big\}.$$

This inequality holds for all $\lambda\in\mathbb{R}$, so we get the desired inequality. In particular, if we choose $\lambda=0$, then

$$dw_A^2(T)\leqslant \frac{1}{2}\sup_{\theta\in\mathbb{R}}\Big\{\Big\|\cos\theta\ Re_A(T)+T^{\sharp_A}T+\sin\theta\ Im_A(T)\Big\|_A^2$$
$$+\Big\|\cos\theta\ Re_A(T)-T^{\sharp_A}T+\sin\theta\ Im_A(T)\Big\|_A^2\Big\}.\quad\square$$

Our next result reads as:

THEOREM 2.18. *Let $T\in\mathscr{B}_A(\mathscr{H})$. Then*

$$dw_A^2(T)\leqslant \inf_{\lambda\in\mathbb{C}}\Big\{\Big(2\|Re(\lambda)\ Re_A(T)+Im(\lambda)\ Im_A(T)\|_A+\Big\|T^{\sharp_A}T-2Re_A(\overline{\lambda}T)\Big\|_A\Big)^2$$
$$+2\|Re_A(\overline{\lambda}T)\|_A-|\lambda|^2+w_A^2(T-\lambda I)\Big\}.$$

In particular, $dw_A(T)\leqslant\sqrt{w_A^2(T)+\|T\|_A^4}$.

Proof. Let $x\in\mathscr{H}$ with $\|x\|_A=1$. Let $\lambda\in\mathbb{C}$. Using Lemma 2.16 we get,

$$\|Tx\|_A^2\|x\|_A^2-|\langle Tx,x\rangle_A|^2=\|Tx-\lambda x\|_A^2\|x\|_A^2-|\langle Tx-\lambda x,x\rangle_A|^2.$$

Using Cartesian decomposition of T, i.e., $T=Re_A(T)+\mathrm{i}\ Im_A(T)$, we get,

$$\begin{aligned}\|Tx\|_A^2&=(\langle Re_A(T)x,x\rangle_A)^2-(\langle Re_A(T-\lambda I)x,x\rangle_A)^2+(\langle Im_A(T)x,x\rangle_A)^2\\&\quad-(\langle Im_A(T-\lambda I)x,x\rangle_A)^2+\|Tx-\lambda x\|_A^2\\&=\langle(2Re_A(T)-Re(\lambda)I)x,x\rangle_A\langle Re(\lambda)x,x\rangle_A\\&\quad+\langle(2Im_A(T)-Im(\lambda)I)x,x\rangle\langle Im(\lambda)x,x\rangle_A+\|Tx-\lambda x\|_A^2\end{aligned}$$

$$
\begin{aligned}
&= 2Re(\lambda)\langle Re_A(T)x,x\rangle_A + 2Im(\lambda)\langle Im_A(T)x,x\rangle_A \\
&\quad -(Re(\lambda))^2 - (Im(\lambda))^2 + \|Tx-\lambda x\|_A^2 \\
&= 2\left(Re(\lambda)\langle Re_A(T)x,x\rangle + Im(\lambda)\langle Im_A(T)x,x\rangle_A\right) - |\lambda|^2 \\
&\quad + \langle Tx-\lambda x, Tx-\lambda x\rangle_A \\
&= 2\left(Re(\lambda)\langle Re_A(T)x,x\rangle_A + Im(\lambda)\langle Im_A(T)x,x\rangle_A\right) \\
&\quad + \left\langle (T^{\sharp_A}T - 2Re_A(\overline{\lambda}T))x,x\right\rangle_A \\
&\leqslant 2\left\|Re(\lambda)\, Re_A(T) + Im(\lambda)\, Im_A(T)\right\|_A + \left\|T^{\sharp_A}T - 2Re_A(\overline{\lambda}T)\right\|_A.
\end{aligned}
$$

Again using Lemma 2.16 we get,

$$
\begin{aligned}
|\langle Tx,x\rangle_A|^2 &= \|Tx\|_A^2 - \|Tx-\lambda x\|_A^2 + |\langle Tx-\lambda x,x\rangle_A|^2 \\
&= 2\langle Re(\overline{\lambda}T)x,x\rangle_A - |\lambda|^2 + |\langle Tx-\lambda x,x\rangle_A|^2 \\
&\leqslant 2\|Re_A(\overline{\lambda}T)\| - |\lambda|^2 + w_A^2(T-\lambda I).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|\langle Tx,x\rangle_A|^2 + \|Tx\|_A^4 \\
&\leqslant 2\|Re_A(\overline{\lambda}T)\| - |\lambda|^2 + w_A^2(T-\lambda I) \\
&\quad + \left(2\left\|Re(\lambda)\, Re_A(T) + Im(\lambda)\, Im_A(T)\right\| + \left\|T^{\sharp_A}T - 2Re_A(\overline{\lambda}T)\right\|_A\right)^2.
\end{aligned}
$$

Therefore, taking supremum over all A-unit vectors in $\mathscr{H}$, and then taking infimum over all $\lambda \in \mathbb{C}$, we get

$$
\begin{aligned}
dw_A^2(T) \leqslant \inf_{\lambda\in\mathbb{C}} \Big\{ &\left(2\left\|Re(\lambda)\, Re_A(T) + Im(\lambda)\, Im_A(T)\right\|_A + \left\|T^{\sharp_A}T - 2Re_A(\overline{\lambda}T)\right\|_A\right)^2 \\
&+ 2\|Re_A(\overline{\lambda}T)\|_A - |\lambda|^2 + w_A^2(T-\lambda I)\Big\}.
\end{aligned}
$$

Taking $\lambda = 0$, we get $dw_A(T) \leqslant \sqrt{w_A^2(T) + \|T\|_A^4}$. □

REMARK 2.19. We would like to note that the inequality in [6, Th. 2.5] follows from Theorem 2.18 by considering $A = I$.

In the following theorem we obtain an upper bound for the A-Davis-Wielandt radius of sum of two operators in $\mathscr{B}_A(\mathscr{H})$.

THEOREM 2.20. *Let $X, Y \in \mathscr{B}_A(\mathscr{H})$. Then*

$$dw_A(X+Y) \leqslant dw_A(X) + dw_A(Y) + w_A(X^{\sharp_A}Y + Y^{\sharp_A}X).$$

In particular, if $A(X^{\sharp_A}Y + Y^{\sharp_A}X) = O$ then

$$dw_A(X+Y) \leqslant dw_A(X) + dw_A(Y).$$

Proof. From the definition of the A-Davis-Wielandt shell we get,

$$\begin{aligned} DW_A(X+Y) &= \Big\{ \Big(\langle (X+Y)x, x \rangle_A, \langle (X+Y)x, (X+Y)x \rangle_A \Big) : x \in \mathscr{H}, \|x\|_A = 1 \Big\} \\ &= \Big\{ \Big(\langle Xx, x \rangle_A, \langle Xx, Xx \rangle_A \Big) + \Big(\langle Yx, x \rangle_A, \langle Yx, Yx \rangle_A \Big) \\ &\quad + \Big(0, \langle (X^{\sharp_A}Y + Y^{\sharp_A}X)x, x \rangle_A \Big) : x \in \mathscr{H}, \|x\|_A = 1 \Big\}. \end{aligned}$$

Hence, $DW_A(X+Y) \subseteq DW_A(X) + DW_A(Y) + L$, where

$$L = \Big\{ \Big(0, \langle (X^{\sharp_A}Y + Y^{\sharp_A}X)x, x \rangle_A \Big) : x \in \mathscr{H}, \|x\|_A = 1 \Big\}.$$

This implies the first inequality of the theorem. In particular, if we consider $A(X^{\sharp_A}Y + Y^{\sharp_A}X) = O$, then we get the second inequality. □

REMARK 2.21. If we consider $A = I$ in Theorem 2.20 then we get the inequalities in [6, Th. 2.6 and Cor. 2.2].

Next we state the following lemma, proof of which can be found in [8, Lemma 3.1].

LEMMA 2.22. *Let $T_{ij} \in \mathscr{B}_A(\mathscr{H})$, for $i, j = 1, 2$. Then $(T_{ij})_{2\times 2} \in \mathscr{B}_{\mathbb{A}}(\mathscr{H} \oplus \mathscr{H})$ and*

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}^{\sharp_{\mathbb{A}}} = \begin{pmatrix} T_{11}^{\sharp_A} & T_{21}^{\sharp_A} \\ T_{12}^{\sharp_A} & T_{22}^{\sharp_A} \end{pmatrix}.$$

Using Theorem 2.20 and Lemma 2.22, we prove the following inequality.

COROLLARY 2.23. Let $X, Y \in \mathscr{B}_A(\mathscr{H})$, then

$$dw_{\mathbb{A}} \begin{pmatrix} O & X \\ Y & O \end{pmatrix} \leqslant \sqrt{\frac{1}{4}\|X\|_A^2 + \|X\|_A^4} + \sqrt{\frac{1}{4}\|Y\|_A^2 + \|Y\|_A^4}.$$

Proof. Clearly, $\begin{pmatrix} O & X \\ O & O \end{pmatrix}^{\sharp_{\mathbb{A}}} \begin{pmatrix} O & O \\ Y & O \end{pmatrix} + \begin{pmatrix} O & O \\ Y & O \end{pmatrix}^{\sharp_{\mathbb{A}}} \begin{pmatrix} O & X \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix}$. Therefore, from Theorem 2.20, we get,

$$\begin{aligned} & dw_{\mathbb{A}} \begin{pmatrix} O & X \\ Y & O \end{pmatrix} \\ & \leqslant dw_{\mathbb{A}} \begin{pmatrix} O & X \\ O & O \end{pmatrix} + dw_{\mathbb{A}} \begin{pmatrix} O & O \\ Y & O \end{pmatrix} \\ & \leqslant \sqrt{w_{\mathbb{A}}^2 \begin{pmatrix} O & X \\ O & O \end{pmatrix} + \left\| \begin{pmatrix} O & X \\ O & O \end{pmatrix} \right\|_{\mathbb{A}}^4} + \sqrt{w_{\mathbb{A}}^2 \begin{pmatrix} O & O \\ Y & O \end{pmatrix} + \left\| \begin{pmatrix} O & O \\ Y & O \end{pmatrix} \right\|_{\mathbb{A}}^4} \end{aligned}$$

$$= \sqrt{\frac{1}{4}\left\|\begin{pmatrix} O & X \\ O & O \end{pmatrix}\right\|_{\mathbb{A}}^2 + \left\|\begin{pmatrix} O & X \\ O & O \end{pmatrix}\right\|_{\mathbb{A}}^4} + \sqrt{\frac{1}{4}\left\|\begin{pmatrix} O & O \\ Y & O \end{pmatrix}\right\|_{\mathbb{A}}^2 + \left\|\begin{pmatrix} O & O \\ Y & O \end{pmatrix}\right\|_{\mathbb{A}}^4},$$

$$\text{as } \mathbb{A}\begin{pmatrix} O & X \\ O & O \end{pmatrix}^2 = \mathbb{A}\begin{pmatrix} O & O \\ Y & O \end{pmatrix}^2 = \begin{pmatrix} O & O \\ O & O \end{pmatrix}, \text{ see [14, Cor. 2.2]}$$

$$= \sqrt{\frac{1}{4}\|X\|_A^2 + \|X\|_A^4} + \sqrt{\frac{1}{4}\|Y\|_A^2 + \|Y\|_A^4}, \text{ by using [7, Remark 3].} \quad \square$$

Our next result reads as:

THEOREM 2.24. *Let $X, Y \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$. Then*

$$dw_{\mathbb{A}}\begin{pmatrix} X & O \\ O & Y \end{pmatrix} = \max\left\{dw_A(X), dw_A(Y)\right\}.$$

Proof. Let $T = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$. Let x be an A-unit vector in $\mathscr{H}$ and let $\tilde{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathscr{H} \oplus \mathscr{H}$. Clearly $\|\tilde{x}\|_{\mathbb{A}} = 1$. So

$$|\langle Xx, x\rangle_A|^2 + \|Xx\|_A^4 = |\langle T\tilde{x}, \tilde{x}\rangle_{\mathbb{A}}|^2 + \|T\tilde{x}\|_{\mathbb{A}}^4 \leqslant dw_{\mathbb{A}}^2(T).$$

Taking supremum over all A-unit vectors in $\mathscr{H}$, we get $dw_A^2(X) \leqslant dw_{\mathbb{A}}^2(T)$. Similarly, we can prove that, $dw_A^2(Y) \leqslant dw_{\mathbb{A}}^2(T)$. Combining above two inequalities, we get

$$\max\{dw_A(X), dw_A(Y)\} \leqslant dw_{\mathbb{A}}(T).$$

To complete the proof, we only need to show $dw_{\mathbb{A}}(T) \leqslant \max\{dw_A(X), dw_A(Y)\}$. Let $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathscr{H} \oplus \mathscr{H}$ be such that $\|z\|_{\mathbb{A}} = 1$, i.e., $\|x\|_A^2 + \|y\|_A^2 = 1$. Then

$$\begin{aligned}
&|\langle Tz, z\rangle|_{\mathbb{A}}^2 + \|Tz\|_{\mathbb{A}}^4 \\
&= |\langle Xx, x\rangle_A + \langle Yy, y\rangle|_A^2 + (\|Xx\|_A^2 + \|Yy\|_A^2)^2 \\
&\leqslant (|\langle Xx, x\rangle_A| + |\langle Yy, y\rangle_A|)^2 + (\|Xx\|_A^2 + \|Yy\|_A^2)^2 \\
&\leqslant \left(\sqrt{|\langle Xx, x\rangle|_A^2 + \|Xx\|_A^4} + \sqrt{|\langle Yy, y\rangle_A|^2 + \|Yy\|_A^4}\right)^2, \text{ by Minkowski inequality} \\
&\leqslant \left(dw_A(X)\|x\|_A^2 + dw_A(Y)\|y\|_A^2\right)^2 \\
&\leqslant \max\{dw_A^2(X), dw_A^2(Y)\}.
\end{aligned}$$

Taking supremum over all $\mathbb{A}$-unit vectors in $\mathscr{H} \oplus \mathscr{H}$, we get

$$dw_{\mathbb{A}}^2(T) \leqslant \max\{dw_A^2(X), dw_A^2(Y)\}, \text{ i.e., } dw_{\mathbb{A}}(T) \leqslant \max\{dw_A(X), dw_A(Y)\}. \quad \square$$

REMARK 2.25. Let $\mathbb{S} = \begin{pmatrix} X & O \\ O & O \end{pmatrix}$ or $\begin{pmatrix} O & O \\ O & X \end{pmatrix}$, where $X \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$. Then by Theorem 2.24 we have, $dw_{\mathbb{A}}(\mathbb{S}) = dw_A(X)$.

Now we prove an important result $dw_A(T) = dw_A(T^{\sharp_A})$ for $T \in \mathscr{B}_A(\mathscr{H})$. For this purpose we need the following arguments. The semi-inner product $\langle \cdot, \cdot \rangle_A$ induces an inner product on the quotient space $\mathscr{H}/\mathscr{N}(A)$ defined as

$$[\overline{x}, \overline{y}] = \langle Ax, y \rangle,$$

for all $\overline{x} = x + \mathscr{N}(A), \overline{y} = y + \mathscr{N}(A) \in \mathscr{H}/\mathscr{N}(A)$. Note that $(\mathscr{H}/\mathscr{N}(A), [\cdot, \cdot])$ is not complete unless $\mathscr{R}(A)$ is closed in $\mathscr{H}$. L. de Branges and J. Rovnyak [11] showed that the completion of $\mathscr{H}/\mathscr{N}(A)$ is isometrically isomorphic to the Hilbert space $\mathscr{R}(A^{1/2})$ with the inner product

$$(A^{1/2}x, A^{1/2}y) = \langle P_A x, P_A y \rangle, \ \forall x, y \in \mathscr{H}.$$

The Hilbert space $\big(\mathscr{R}(A^{1/2}), (\cdot, \cdot)\big)$ is denoted by $\mathbf{R}(A^{1/2})$, and we use $\| \cdot \|_{\mathbf{R}(A^{1/2})}$ to represent the norm induced by the inner product $(\cdot, \cdot)$. For more information related to the Hilbert space $\mathbf{R}(A^{1/2})$, we refer the interested readers to [1]. Note that the fact $\mathscr{R}(A) \subseteq \mathscr{R}(A^{1/2})$ implies that $(Ax, Ay) = \langle x, y \rangle_A$. This implies the useful relation

$$\|Ax\|_{\mathbf{R}(A^{1/2})} = \|x\|_A, \ \forall x \in \mathscr{H}.$$

To proceed further we need the following lemma which gives a nice connection between $T \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$ and $\widetilde{T} \in \mathscr{B}(\mathbf{R}(A^{1/2}))$.

LEMMA 2.26. ([1, Prop. 3.6]) *Let $T \in \mathscr{B}(\mathscr{H})$ and let $Z_A : \mathscr{H} \to \mathbf{R}(A^{1/2})$ be defined by $Z_A x = Ax$, $\forall\ x \in \mathscr{H}$. Then $T \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$ if and only if there exists unique $\widetilde{T} \in \mathscr{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A T = \widetilde{T} Z_A$.*

There are many important well-known relations between T and $\widetilde{T}$, we mention a few of them in the form of the following lemma.

LEMMA 2.27. ([21, Prop. 2.9]) *Let $T \in \mathscr{B}_A(\mathscr{H})$. Then*

$$\widetilde{T^{\sharp_A}} = \big(\widetilde{T}\big)^* \ \text{and} \ \widetilde{(T^{\sharp_A})^{\sharp_A}} = \widetilde{T}.$$

We now prove the following proposition.

PROPOSITION 2.28. *Let $T \in \mathscr{B}_A(\mathscr{H})$. Then*

$$dw_A(T) = dw_A(T^{\sharp_A}).$$

Proof. It follows from [17, Lemma 2] that $dw_A(T) = dw(\widetilde{T})$. Since $\widetilde{T} \in \mathscr{B}(\mathbf{R}(A^{1/2}))$ and $\mathbf{R}(A^{1/2})$ is a complex Hilbert space, so from [19, Th. 3.3 (c)] we have, $dw(\widetilde{T}) = dw((\widetilde{T})^*)$. Hence, we have from Lemma 2.27 that $dw(\widetilde{T}) = dw(\widetilde{T^{\sharp_A}})$. Thus, $dw(\widetilde{T}) = dw_A(T^{\sharp_A})$. This completes the proof. $\square$

By using Proposition 2.28 we prove the following lemma.

LEMMA 2.29. *Let $T \in \mathscr{B}_A(\mathscr{H})$. Then, $dw_A(U^{\sharp_A}TU) = dw_A(T)$, for every A-unitary operator $U \in \mathscr{B}_A(\mathscr{H})$.*

Proof. Let $U \in \mathscr{B}_A(\mathscr{H})$ be an A-unitary operator. Let $(\lambda,\mu) \in DW_A(U^{\sharp_A}TU)$. Then there exists $x \in \mathscr{H}$ with $\|x\|_A = 1$ such that $\lambda = \langle U^{\sharp_A}TUx, x\rangle_A$ and $\mu = \|U^{\sharp_A}TUx\|_A^2$. It is easy to verify that $\lambda = \langle TUx, Ux\rangle_A$ and $\mu = \|TUx\|_A^2$. Since $\|Ux\|_A = 1$, so $(\lambda,\mu) \in DW_A(T)$. Hence, $DW_A(U^{\sharp_A}TU) \subseteq DW_A(T)$. This implies that $dw_A(U^{\sharp_A}TU) \leqslant dw_A(T)$. Next we prove that $DW_A(T^{\sharp_A}) \subseteq DW_A\left((U^{\sharp_A}TU)^{\sharp_A}\right)$. Let $(\beta,\gamma) \in DW_A(T^{\sharp_A})$. Then there exists $x \in \mathscr{H}$ with $\|x\|_A = 1$ such that $\beta = \langle T^{\sharp_A}x, x\rangle_A$ and $\gamma = \|T^{\sharp_A}x\|_A^2$. Now x can be written as $x = P_Ax + y$, where $y \in \mathscr{N}(A)$. We have,

$$\begin{aligned}
\beta &= \langle T^{\sharp_A}x, x\rangle_A = \langle T^{\sharp_A}(P_Ax+y), (P_Ax+y)\rangle_A \\
&= \langle T^{\sharp_A}P_Ax, P_Ax\rangle_A, \quad T^{\sharp_A}(\mathscr{N}(A)) \subseteq \mathscr{N}(A) \\
&= \langle T^{\sharp_A}(U^{\sharp_A})^{\sharp_A}U^{\sharp_A}x, (U^{\sharp_A})^{\sharp_A}U^{\sharp_A}x\rangle_A \\
&= \langle U^{\sharp_A}T^{\sharp_A}(U^{\sharp_A})^{\sharp_A}U^{\sharp_A}x, U^{\sharp_A}x\rangle_A \\
&= \langle (U^{\sharp_A}TU)^{\sharp_A}U^{\sharp_A}x, U^{\sharp_A}x\rangle_A
\end{aligned}$$

and

$$\begin{aligned}
\gamma &= \langle T^{\sharp_A}x, T^{\sharp_A}x\rangle_A = \langle U^{\sharp_A}T^{\sharp_A}x, U^{\sharp_A}T^{\sharp_A}x\rangle_A \\
&= \langle U^{\sharp_A}T^{\sharp_A}(P_Ax+y), U^{\sharp_A}T^{\sharp_A}(P_Ax+y)\rangle_A \\
&= \langle U^{\sharp_A}T^{\sharp_A}P_Ax, U^{\sharp_A}T^{\sharp_A}P_Ax\rangle_A, \quad T^{\sharp_A}(\mathscr{N}(A)) \subseteq \mathscr{N}(A) \\
&= \langle U^{\sharp_A}T^{\sharp_A}(U^{\sharp_A})^{\sharp_A}U^{\sharp_A}x, U^{\sharp_A}T^{\sharp_A}(U^{\sharp_A})^{\sharp_A}U^{\sharp_A}x\rangle_A \\
&= \|(U^{\sharp_A}TU)^{\sharp_A}U^{\sharp_A}x\|_A^2.
\end{aligned}$$

Since $\|U^{\sharp_A}x\|_A = 1$, so $(\beta,\gamma) \in DW_A\left((U^{\sharp_A}TU)^{\sharp_A}\right)$.

Hence, $DW_A(T^{\sharp_A}) \subseteq DW_A\left((U^{\sharp_A}TU)^{\sharp_A}\right)$, and so $dw_A(T^{\sharp_A}) \leqslant dw_A\left((U^{\sharp_A}TU)^{\sharp_A}\right)$. Thus, it follows from Proposition 2.28 that $dw_A(T) \leqslant dw_A\left(U^{\sharp_A}TU\right)$. Hence, $dw_A(U^{\sharp_A}TU) = dw_A(T)$. $\square$

Now by using Lemma 2.29, we prove the following lemma.

LEMMA 2.30. *Let $X, Y \in \mathscr{B}_A(\mathscr{H})$. Then*

(a) $dw_{\mathbb{A}}\begin{pmatrix} O & X \\ e^{\mathrm{i}\theta}Y & O \end{pmatrix} = dw_{\mathbb{A}}\begin{pmatrix} O & X \\ Y & O \end{pmatrix}$, *for every $\theta \in \mathbb{R}$.*

(b) $dw_{\mathbb{A}}\begin{pmatrix} O & X \\ Y & O \end{pmatrix} = dw_{\mathbb{A}}\begin{pmatrix} O & Y \\ X & O \end{pmatrix}$.

Proof.

(a) Let $U = \begin{pmatrix} I & O \\ O & e^{\mathrm{i}\frac{\theta}{2}}I \end{pmatrix}$ and let $x = (x_1, x_2) \in \mathscr{H} \oplus \mathscr{H}$. It is easy to see that $\|Ux\|_{\mathbb{A}} = \|U^{\sharp_{\mathbb{A}}}x\|_{\mathbb{A}} = \|x\|_{\mathbb{A}}$. This implies that U is an $\mathbb{A}$-unitary operator. Now,

$U^{\sharp_{\mathbb{A}}} = \begin{pmatrix} P_A & O \\ O & e^{-\mathrm{i}\frac{\theta}{2}} P_A \end{pmatrix}$. Using Lemma 2.29 we get,

$$\begin{aligned} dw_{\mathbb{A}} \begin{pmatrix} O & X \\ e^{\mathrm{i}\theta} Y & O \end{pmatrix} &= dw_{\mathbb{A}} \left(U^{\sharp_{\mathbb{A}}} \begin{pmatrix} O & X \\ e^{\mathrm{i}\theta} Y & O \end{pmatrix} U \right) \\ &= dw_{\mathbb{A}} \left(\begin{pmatrix} P_A & O \\ O & P_A \end{pmatrix} \begin{pmatrix} O & e^{\mathrm{i}\frac{\theta}{2}} X \\ e^{\mathrm{i}\frac{\theta}{2}} Y & O \end{pmatrix} \right) \\ &= dw_{\mathbb{A}} \begin{pmatrix} O & e^{\mathrm{i}\frac{\theta}{2}} X \\ e^{\mathrm{i}\frac{\theta}{2}} Y & O \end{pmatrix} \\ &= dw_{\mathbb{A}} \begin{pmatrix} O & X \\ Y & O \end{pmatrix}. \end{aligned}$$

(b) Considering $U = \begin{pmatrix} O & I \\ I & O \end{pmatrix}$. Clearly, U is an $\mathbb{A}$-unitary operator. Similar as above, using Lemma 2.29, we get (b). □

By using Lemma 2.30, we obtain an upper bound for the A-Davis-Wielandt radius of sum of product operators in $\mathscr{B}_A(\mathscr{H})$.

THEOREM 2.31. *Let $P,Q,X,Y \in \mathscr{B}_A(\mathscr{H})$. Then for any $t \in \mathbb{R} \setminus \{0\}$, we have*

$$dw_A^2(PXQ^{\sharp_A} \pm QYP^{\sharp_A}) \leqslant \left(t^2 \|P\|_A^2 + \frac{1}{t^2} \|Q\|_A^2 \right)^2 \left\{ \left(t^2 \|PX\|_A^2 + \frac{1}{t^2} \|QY\|_A^2 \right)^2 + \alpha^2 \right\},$$

where $\alpha = w_{\mathbb{A}} \begin{pmatrix} O & X \\ Y & O \end{pmatrix}$.

Proof. Let $C, Z \in \mathscr{B}_{\mathbb{A}}(\mathscr{H} \oplus \mathscr{H})$ be such that $C = \begin{pmatrix} P & Q \\ O & O \end{pmatrix}$ and $Z = \begin{pmatrix} O & X \\ Y & O \end{pmatrix}$. Then we have, $CZC^{\sharp_{\mathbb{A}}} = \begin{pmatrix} PXQ^{\sharp_A} + QYP^{\sharp_A} & O \\ O & O \end{pmatrix}$. Therefore,

$$\begin{aligned} dw_A^2(PXQ^{\sharp_A} + QYP^{\sharp_A}) &= dw_{\mathbb{A}}^2 \begin{pmatrix} PXQ^{\sharp_A} + QYP^{\sharp_A} & O \\ O & O \end{pmatrix} \\ &= dw_{\mathbb{A}}^2 (CZC^{\sharp_{\mathbb{A}}}) \\ &= \sup_{\|x\|_{\mathbb{A}}=1} \left\{ |\langle CZC^{\sharp_{\mathbb{A}}} x, x \rangle_{\mathbb{A}}|^2 + \|CZC^{\sharp_{\mathbb{A}}} x\|_{\mathbb{A}}^4 \right\} \\ &= \sup_{\|x\|_{\mathbb{A}}=1} \left\{ |\langle ZC^{\sharp_{\mathbb{A}}} x, C^{\sharp_{\mathbb{A}}} x \rangle_{\mathbb{A}}|^2 + \|CZC^{\sharp_{\mathbb{A}}} x\|_{\mathbb{A}}^4 \right\} \\ &\leqslant \sup_{\|x\|_{\mathbb{A}}=1} \left\{ w_{\mathbb{A}}^2(Z) \|C^{\sharp_{\mathbb{A}}} x\|_{\mathbb{A}}^4 + \|CZ\|_{\mathbb{A}}^4 \|C^{\sharp_{\mathbb{A}}} x\|_{\mathbb{A}}^4 \right\} \\ &= \left(w_{\mathbb{A}}^2(Z) + \|CZ\|_{\mathbb{A}}^4 \right) \|C\|_{\mathbb{A}}^4. \end{aligned}$$

It is easy to see that $\|C\|_{\mathbb{A}}^2 = \|PP^{\sharp_A} + QQ^{\sharp_A}\|_A$ and $\|CZ\|_{\mathbb{A}}^2 = \|(QY)(QY)^{\sharp_A} + (PX)(PX)^{\sharp_A}\|_A$. Therefore, from the above inequality, we get

$$dw_A^2(PXQ^{\sharp_A} + QYP^{\sharp_A}) \leqslant \left(\|P\|_A^2 + \|Q\|_A^2\right)^2 \left\{(\|QY\|_A^2 + \|PX\|_A^2)^2 + w_{\mathbb{A}}^2(Z)\right\}.$$

Replacing Y by $-Y$ in the above inequality and using Lemma 2.30 (a), we get

$$dw_A^2(PXQ^{\sharp_A} - QYP^{\sharp_A}) \leqslant \left(\|P\|_A^2 + \|Q\|_A^2\right)^2 \left\{(\|QY\|_A^2 + \|PX\|_A^2)^2 + w_{\mathbb{A}}^2(Z)\right\}.$$

Clearly, the above two inequalities hold for all $P,Q \in \mathscr{B}_A(\mathscr{H})$. So, replacing P by tP and Q by $\frac{1}{t}Q$, we get the required inequality of the theorem. □

COROLLARY 2.32. Let $P,Q,X,Y \in \mathscr{B}_A(\mathscr{H})$ with $\|P\|_A, \|Q\|_A \neq 0$. Then

$$(i)\ dw_A^2(PXQ^{\sharp_A} \pm QYP^{\sharp_A}) \leqslant 4\|P\|_A^2\|Q\|_A^2\left\{\left(\frac{\|P\|_A}{\|Q\|_A}\|QY\|_A^2 + \frac{\|Q\|_A}{\|P\|_A}\|PX\|_A^2\right)^2 + \alpha^2\right\},$$

where $\alpha = w_{\mathbb{A}}\begin{pmatrix} O & X \\ Y & O \end{pmatrix}$.

$$(ii)\ dw_A^2(X \pm Y) \leqslant 4\left\{\left(\|X\|_A^2 + \|Y\|_A^2\right)^2 + w_{\mathbb{A}}^2\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\right\}.$$

Proof. Considering $t = \sqrt{\frac{\|Q\|_A}{\|P\|_A}}$ in Theorem 2.31, we get the inequality (i). Choosing $P = Q = I$ in (i), we get the inequality (ii). □

COROLLARY 2.33. Let $P,Q,X,Y \in \mathscr{B}_A(\mathscr{H})$ be such that $\|PX\|_A, \|QY\|_A \neq 0$. Then

$$(i)\ dw_A^2(PXQ^{\sharp_A} \pm QYP^{\sharp_A}) \leqslant \left(\frac{\|QY\|_A}{\|PX\|_A}\|P\|_A^2 + \frac{\|PX\|_A}{\|QY\|_A}\|Q\|_A^2\right)^2 \left\{4\|PX\|_A^2\|QY\|_A^2 + \alpha^2\right\},$$

where $\alpha = w_{\mathbb{A}}\begin{pmatrix} O & X \\ Y & O \end{pmatrix}$.

$$(ii)\ dw_A^2(X \pm Y) \leqslant \left(\frac{\|Y\|_A}{\|X\|_A} + \frac{\|X\|_A}{\|Y\|_A}\right)^2 \left\{(2\|X\|_A\|Y\|_A)^2 + w_{\mathbb{A}}^2\begin{pmatrix} O & X \\ Y & O \end{pmatrix}\right\}.$$

Proof. Considering $t = \sqrt{\frac{\|QY\|_A}{\|PX\|_A}}$ in Theorem 2.31, we get the inequality (i). Choosing $P = Q = I$ in (i), we get the inequality (ii). □

REMARK 2.34. Feki in [17, Prop. 3] proved that if $X,Y \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$ then the following inequality holds:

$$dw_A^2(X+Y) \leqslant 2\Big(dw_A(X) + dw_A(Y)\Big) + 4\Big(dw_A(X) + dw_A(Y)\Big)^2.$$

If we consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then [17, Prop. 3] gives $dw_A(X+Y) \leqslant 4.2994$, whereas Theorem 2.20 gives $dw_A(X+Y) \leqslant 2.621320$, Corollary 2.32 (ii) gives $dw_A(X+Y) \leqslant 3.240466$ and Corollary 2.33 (ii) gives $dw_A(X+Y) \leqslant 3.26928$. Thus the bounds obtained in Theorem 2.20, Corollary 2.32 (ii) and Corollary 2.33 (ii) are better than that obtained in [17, Prop. 3].

Proceeding similarly as in Theorem 2.31 we can prove the following results.

THEOREM 2.35. *Let $P,Q,X,Y \in \mathscr{B}_A(\mathscr{H})$. Then for any $t \in \mathbb{R} \setminus \{0\}$, we have*

$$dw_A^2(PXQ^{\sharp_A} \pm QYP^{\sharp_A}) \leqslant \left(t^2\|P\|_A^2 + \frac{1}{t^2}\|Q\|_A^2\right)^2 \left\{\left(t^2\|YP^{\sharp_A}\|_A^2 + \frac{1}{t^2}\|XQ^{\sharp_A}\|_A^2\right)^2 + \alpha^2\right\},$$

$$dw_A^2(P^{\sharp_A}XQ \pm Q^{\sharp_A}YP) \leqslant \left(t^2\|P\|_A^2 + \frac{1}{t^2}\|Q\|_A^2\right)^2 \left\{\left(t^2\|YP\|_A^2 + \frac{1}{t^2}\|XQ\|_A^2\right)^2 + \alpha^2\right\}$$

and

$$dw_A^2(P^{\sharp_A}XQ \pm Q^{\sharp_A}YP) \leqslant \left(t^2\|P\|_A^2 + \frac{1}{t^2}\|Q\|_A^2\right)^2 \left\{\left(t^2\|P^{\sharp_A}X\|_A^2 + \frac{1}{t^2}\|Q^{\sharp_A}Y\|_A^2\right)^2 + \alpha^2\right\},$$

where $\alpha = w_{\mathbb{A}}\begin{pmatrix} O & X \\ Y & O \end{pmatrix}$.

Now we determine the exact value of the $\mathbb{A}$-Davis-Wielandt radius of special type of 2×2 operator matrices in $\mathscr{B}_{A^{1/2}}(\mathscr{H} \oplus \mathscr{H})$.

THEOREM 2.36. *Let $X \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$ and $\mathbb{T} = \begin{pmatrix} I & X \\ O & O \end{pmatrix}$. Then*

$$dw_{\mathbb{A}}(\mathbb{T}) = \begin{cases} \sqrt{2}, & \|X\|_A = 0 \\ (\cos\theta_0 + \|X\|_A \sin\theta_0)(\cos^2\theta_0 + (\cos\theta_0 + \|X\|_A \sin\theta_0)^2)^{\frac{1}{2}}, & \|X\|_A \neq 0, \end{cases}$$

where $b = \|X\|_A$, $p = -\frac{2b^2-5}{2b}$, $q = -\frac{2b^2-2}{b^2}$, $r = -\frac{3}{2b}$, $s = \frac{1}{2^4 3^3 b^6}(8b^8 + 20b^6 + 45b^4 + 61b^2 + 28)$, $\alpha = \frac{1}{27}(2p^3 - 9pq + 27r)$, $\beta = (-\frac{\alpha}{2} + \sqrt{s})^{\frac{1}{3}}$, $\gamma = (-\frac{\alpha}{2} - \sqrt{s})^{\frac{1}{3}}$ *and* $\theta_0 = \tan^{-1}(\beta + \gamma - \frac{p}{3})$.

Proof. Let $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathscr{H} \oplus \mathscr{H}$ be such that $\|z\|_{\mathbb{A}} = 1$, i.e, $\|x\|_A^2 + \|y\|_A^2 = 1$. Then $\langle \mathbb{T}z, z\rangle_{\mathbb{A}} = \langle x + Xy, x\rangle_A$ and $\langle \mathbb{T}z, \mathbb{T}z\rangle_{\mathbb{A}} = \langle x + Xy, x + Xy\rangle_A$. Now, we have

$$\begin{aligned} &|\langle \mathbb{T}z, z\rangle_{\mathbb{A}}|^2 + |\langle \mathbb{T}z, \mathbb{T}z\rangle_{\mathbb{A}}|^2 \\ &\leqslant \|x + Xy\|_A^2 \|x\|_A^2 + \|x + Xy\|_A^4 \\ &= \|x + Xy\|_A^2 \left(\|x\|_A^2 + \|x + Xy\|_A^2\right) \end{aligned}$$

$$\leqslant \sup_{\|x\|_A^2+\|y\|_A^2=1} (\|x\|_A+\|X\|_A\|y\|_A)^2(\|x\|_A^2+(\|x\|_A+\|X\|_A\|y\|_A)^2)$$
$$= \sup_{\theta\in[0,\frac{\pi}{2}]} (\cos\theta+\|X\|_A\sin\theta)^2(\cos^2\theta+(\cos\theta+\|X\|_A\sin\theta)^2).$$

First we consider the case $\|X\|_A=0$. Then

$$\sup_{\theta\in[0,\frac{\pi}{2}]} (\cos\theta+\|X\|_A\sin\theta)^2(\cos^2\theta+(\cos\theta+\|X\|_A\sin\theta)^2)=2.$$

Therefore, $dw_{\mathbb{A}}(\mathbb{T})\leqslant\sqrt{2}$. Now let $z=\begin{pmatrix}x\\0\end{pmatrix}$ be such that $\|z\|_{\mathbb{A}}=1$, i.e., $\|x\|_A=1$. Then $\langle\mathbb{T}z,z\rangle_{\mathbb{A}}=\|x\|_A^2$ and $\langle\mathbb{T}z,\mathbb{T}z\rangle_{\mathbb{A}}=\|x\|_A^2$. Hence, $(|\langle\mathbb{T}z,z\rangle_{\mathbb{A}}|^2+|\langle\mathbb{T}z,\mathbb{T}z\rangle_{\mathbb{A}}|^2)^{\frac{1}{2}}=\sqrt{2}$. Therefore, $dw_{\mathbb{A}}(\mathbb{T})=\sqrt{2}$.

Next we consider the case $\|X\|_A\neq 0$. Then

$$\sup_{\theta\in[0,\frac{\pi}{2}]} (\cos\theta+\|X\|_A\sin\theta)^2(\cos^2\theta+(\cos\theta+\|X\|_A\sin\theta)^2)$$
$$=(\cos\theta_0+\|X\|_A\sin\theta_0)^2(\cos^2\theta_0+(\cos\theta_0+\|X\|_A\sin\theta_0)^2),$$

where $b=\|X\|_A$, $p=-\frac{2b^2-5}{2b}$, $q=-\frac{2b^2-2}{b^2}$, $r=-\frac{3}{2b}$, $s=\frac{1}{2^4 3^3 b^6}(8b^8+20b^6+45b^4+61b^2+28)$, $\alpha=\frac{1}{27}(2p^3-9pq+27r)$, $\beta=(-\frac{\alpha}{2}+\sqrt{s})^{\frac{1}{3}}$, $\gamma=(-\frac{\alpha}{2}-\sqrt{s})^{\frac{1}{3}}$ and $\theta_0=\tan^{-1}(\beta+\gamma-\frac{p}{3})$. Therefore,

$$dw_{\mathbb{A}}(\mathbb{T})\leqslant(\cos\theta_0+\|X\|_A\sin\theta_0)(\cos^2\theta_0+(\cos\theta_0+\|X\|_A\sin\theta_0)^2)^{\frac{1}{2}}.$$

We now show that there exists a sequence $\{z_n\}$ in $\mathscr{H}\oplus\mathscr{H}$ with $\|z_n\|_{\mathbb{A}}=1$ such that $\lim_{n\to\infty}(|\langle\mathbb{T}z_n,z_n\rangle_{\mathbb{A}}|^2+|\langle\mathbb{T}z_n,\mathbb{T}z_n\rangle_{\mathbb{A}}|^2)^{\frac{1}{2}}=(\cos\theta_0+\|X\|_A\sin\theta_0)(\cos^2\theta_0+(\cos\theta_0+\|X\|_A\sin\theta_0)^2)^{\frac{1}{2}}$. Since $X\in\mathscr{B}_{A^{1/2}}(\mathscr{H})$, there exists a sequence $\{y_n\}$ in $\mathscr{H}$ with $\|y_n\|_A=1$ such that $\lim_{n\to\infty}\|Xy_n\|_A=\|X\|_A$. Let $z_n^k=\frac{1}{\sqrt{\|Xy_n\|_A^2+k^2}}\begin{pmatrix}Xy_n\\ky_n\end{pmatrix}$, where $k\geqslant 0$. Then $|\langle\mathbb{T}z_n^k,z_n^k\rangle_{\mathbb{A}}|^2+|\langle\mathbb{T}z_n^k,\mathbb{T}z_n^k\rangle_{\mathbb{A}}|^2=\frac{(1+k)^2\|Xy_n\|_A^4}{(\|Xy_n\|_A^2+k^2)^2}\left(1+(1+k)^2\right)$

$$=\left(\frac{\|Xy_n\|_A}{\sqrt{\|Xy_n\|_A^2+k^2}}+\frac{k\|Xy_n\|_A}{\sqrt{\|Xy_n\|_A^2+k^2}}\right)^2\left(\frac{\|Xy_n\|_A^2}{\|Xy_n\|_A^2+k^2}+\left(\frac{\|Xy_n\|_A}{\sqrt{\|Xy_n\|_A^2+k^2}}+\frac{k\|Xy_n\|_A}{\sqrt{\|Xy_n\|_A^2+k^2}}\right)^2\right).$$

We can choose $k_0\geqslant 0$ such that $\frac{\|X\|_A}{\sqrt{\|X\|_A^2+k_0^2}}=\cos\theta_0$ and $\frac{k_0}{\sqrt{\|X\|_A^2+k_0^2}}=\sin\theta_0$. Therefore, if we choose $z_n=\frac{1}{\sqrt{\|Xy_n\|_A^2+k_0^2}}\begin{pmatrix}Xy_n\\k_0y_n\end{pmatrix}$, then $\lim_{n\to\infty}(|\langle\mathbb{T}z_n,z_n\rangle_{\mathbb{A}}|^2+|\langle\mathbb{T}z_n,\mathbb{T}z_n\rangle_{\mathbb{A}}|^2)^{\frac{1}{2}}$ $=\left(\cos\theta_0+\|X\|_A\sin\theta_0\right)\left(\cos^2\theta_0+(\cos\theta_0+\|X\|_A\sin\theta_0)^2\right)^{\frac{1}{2}}$. This completes the proof. □

Our final result reads as:

THEOREM 2.37. *Let* $X \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$ *and* $\mathbb{S} = \begin{pmatrix} O & X \\ O & O \end{pmatrix}$. *Then*

$$dw_{\mathbb{A}}(\mathbb{S}) = \begin{cases} 0, & \|X\|_A = 0 \\ \frac{\|X\|_A}{2\sqrt{1-\|X\|_A^2}}, & \|X\|_A < \frac{1}{\sqrt{2}} \\ \|X\|_A^2, & \|X\|_A \geqslant \frac{1}{\sqrt{2}}. \end{cases}$$

Proof. Let $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathscr{H} \oplus \mathscr{H}$ be such that $\|z\|_{\mathbb{A}} = 1$, i.e, $\|x\|_A^2 + \|y\|_A^2 = 1$. Then $\langle \mathbb{S}z, z\rangle_{\mathbb{A}} = \langle Xy, x\rangle_A$ and $\langle \mathbb{S}z, \mathbb{S}z\rangle_{\mathbb{A}} = \langle Xy, Xy\rangle_A$. Now we have

$$\begin{aligned} |\langle \mathbb{S}z, z\rangle_{\mathbb{A}}|^2 + |\langle \mathbb{S}z, \mathbb{S}z\rangle_{\mathbb{A}}|^2 &\leqslant \|Xy\|_A^2\|x\|_A^2 + \|Xy\|_A^4 \\ &\leqslant \sup_{\|x\|_A^2+\|y\|_A^2=1} \left(\|X\|_A^2\|y\|_A^2\|x\|_A^2 + \|X\|_A^4\|y\|_A^4\right) \\ &= \sup_{\theta\in[0,\frac{\pi}{2}]} \|X\|_A^2 \sin^2\theta \left(\cos^2\theta + \|X\|_A^2 \sin^2\theta\right). \end{aligned}$$

First we consider the case $\|X\|_A = 0$. Then it is easy to see that $dw_{\mathbb{A}}(\mathbb{S}) = 0$.

Next we consider the case $0 < \|X\|_A < \frac{1}{\sqrt{2}}$. Then

$$\sup_{\theta\in[0,\frac{\pi}{2}]} \|X\|_A^2 \sin^2\theta \left(\cos^2\theta + \|X\|_A^2 \sin^2\theta\right) = \frac{\|X\|_A^2}{4(1-\|X\|_A^2)}.$$

Therefore, $dw_{\mathbb{A}}(\mathbb{S}) \leqslant \frac{\|X\|_A}{2\sqrt{(1-\|X\|_A^2)}}$. We now show that there exists a sequence $\{z_n\}$ in $\mathscr{H} \oplus \mathscr{H}$ with $\|z_n\|_{\mathbb{A}} = 1$ such that

$$\lim_{n\to\infty} \{|\langle \mathbb{S}z_n, z_n\rangle_{\mathbb{A}}|^2 + |\langle \mathbb{S}z_n, \mathbb{S}z_n\rangle_{\mathbb{A}}|^2\}^{\frac{1}{2}} = \frac{\|X\|_A}{2\sqrt{(1-\|X\|_A^2)}}.$$

Since $X \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$, there exists a sequence $\{y_n\}$ in $\mathscr{H}$ with $\|y_n\|_A = 1$ such that $\lim_{n\to\infty} \|Xy_n\|_A = \|X\|_A$. Let $z_n = \frac{1}{\sqrt{\|Xy_n\|_A^2+k^2}} \begin{pmatrix} Xy_n \\ ky_n \end{pmatrix}$, where $k = \frac{\|X\|_A}{\sqrt{1-2\|X\|_A^2}}$. Then

$$\lim_{n\to\infty} \{|\langle \mathbb{S}z_n, z_n\rangle_{\mathbb{A}}|^2 + |\langle \mathbb{S}z_n, \mathbb{S}z_n\rangle_{\mathbb{A}}|^2\}^{\frac{1}{2}} = \frac{\|X\|_A}{2\sqrt{1-\|X\|_A^2}}.$$

Therefore, $dw_{\mathbb{A}}(\mathbb{S}) = \frac{\|X\|_A}{2\sqrt{(1-\|X\|_A^2)}}$.

Now we consider the case $\|X\|_A \geqslant \frac{1}{\sqrt{2}}$. Then

$$\sup_{\theta\in[0,\frac{\pi}{2}]} \|X\|_A^2 \sin^2\theta \left(\cos^2\theta + \|X\|_A^2 \sin^2\theta\right) = \|X\|_A^4.$$

Therefore, $dw_{\mathbb{A}}(\mathbb{S}) \leqslant \|X\|_A^2$. We now show that there exists a sequence $\{z_n\}$ in $\mathscr{H} \oplus \mathscr{H}$ with $\|z_n\|_{\mathbb{A}} = 1$ such that

$$\lim_{n\to\infty} (|\langle \mathbb{S}z_n, z_n\rangle_{\mathbb{A}}|^2 + |\langle \mathbb{S}z_n, \mathbb{S}z_n\rangle_{\mathbb{A}}|^2)^{\frac{1}{2}} = \|X\|_A^2.$$

Since $X \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$, there exists a sequence $\{y_n\}$ in $\mathscr{H}$ with $\|y_n\|_A = 1$ such that $\lim_{n\to\infty} \|Xy_n\|_A = \|X\|_A$. If we consider $z_n = \begin{pmatrix} 0 \\ y_n \end{pmatrix}$, then $\langle \mathbb{S}z_n, z_n\rangle_{\mathbb{A}} = 0$ and $\langle \mathbb{S}z_n, \mathbb{S}z_n\rangle_{\mathbb{A}} = \|Xy_n\|_A^2$. Therefore, $\lim_{n\to\infty}(|\langle \mathbb{S}z_n, z_n\rangle_{\mathbb{A}}|^2 + |\langle \mathbb{S}z_n, \mathbb{S}z_n\rangle_{\mathbb{A}}|^2)^{\frac{1}{2}} = \|X\|_A^2$. This completes the proof. □

Proceeding similarly as in Theorem 2.37 we also get the following result.

REMARK 2.38. Let $X \in \mathscr{B}_{A^{1/2}}(\mathscr{H})$ and $\mathbb{S} = \begin{pmatrix} O & O \\ X & O \end{pmatrix}$. Then

$$dw_{\mathbb{A}}(\mathbb{S}) = \begin{cases} 0, & \|X\|_A = 0 \\ \frac{\|X\|_A}{2\sqrt{1-\|X\|_A^2}}, & \|X\|_A < \frac{1}{\sqrt{2}} \\ \|X\|_A^2, & \|X\|_A \geqslant \frac{1}{\sqrt{2}}. \end{cases}$$

REMARK 2.39. We note that Theorem 2.36 and Theorem 2.37 generalize the results in [6, Th. 3.1] and [6, Th. 3.2], respectively.

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