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AN EIGENVECTOR–EIGENVALUE–IDENTITY FOR MATRICES WITH A NON–SEMI–SIMPLE EIGENVALUE

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Abstract. We prove an eigenvector-eigenvalue-identity for a complex matrix, concerning some orthonormal basis of the generalized eigenspace associated to an eigenvalue that has exactly one Jordan block in the Jordan canonical form of the matrix. The new formula extends the previous one for the case of simple or semi-simple eigenvalues.

1. Introduction

Recently, the so-called *eigenvector-eigenvalue-identity*

$$|v_{ij}|^2 = \frac{\prod_{k=1}^{n-1}[\lambda_j(A)-\lambda_k(A_i)]}{\prod_{k=1,k\neq j}^{n}[\lambda_j(A)-\lambda_k(A)]}, \tag{1}$$

which was named in [2] and obtained independently several times by different authors for Hermitian matrices and more general normal matrices since the 1960s (see, for example, [5, 6] and the references in the expository paper [2]), has inspired much interest in the literature. The paper [2] surveyed a "surprisingly complicated" history of the development for the eigenvector-eigenvalue-identity in linear algebra, numerical linear algebra and the other areas, and provided some extensions for more general matrices. Inspired by the first proof given in [2] that is based on the concept of the adjugate of a matrix, and with the help of an orthogonal projection technique, the identity has been proved to be true in [4] for a class of diagonalizable matrices, based on a more general result without the original orthogonality assumption of the involved eigenvector to the ones of other eigenvalues.

The formula (1) expresses the modulus square of the i-th component of the j-th eigenvector in a normalized eigenvector basis $v_1,\dots,v_n$ of $\mathbb{C}^n$ in terms of their corresponding eigenvalues $\lambda_1(A),\dots,\lambda_n(A)$ of an $n\times n$ diagonalizable matrix A and the eigenvalues $\lambda_1(A_i),\dots,\lambda_{n-1}(A_i)$ of its sub-matrix A_i that remains after the i-th row and the i-th column are deleted from A.

It is easy to see that modulus squares of components for a single eigenvector cannot be uniquely determined by a mathematical formula in terms of only eigenvalues in

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the case that the associated eigenvalue is not simple, because of the fact that either the corresponding eigenspace is at least two-dimensional or there are generalized eigenvectors for the given eigenvalue. In fact, formula (1) has no meaning when its right-hand side denominator becomes zero, or equivalently, $\lambda_j(A)$ is a multiple eigenvalue.

Can one discover a relation between eigenvalues and modulus squares of eigenvector components for more general matrices? As suggested by Remark 4 of [2], an extended eigenvector-eigenvalue-identity could be obtained with the help of formula (11) therein when the matrix A is non-normal but still diagonalizable and the concerned eigenvalue is simple. An extended eigenvector-eigenvalue formula has been found in [3] when the concerned eigenvalue is semi-simple for a diagonalizable matrix, based on a differentiation analysis of the involved determinant functions. However, as mentioned above, it appears impossible to determine the component modulus square of an individual eigenvector from the formula if the eigenvalue has algebraic multiplicity at least two, but the sum of the modulus squares of the components for all the normalized eigenvectors, which correspond to the same eigenvalue and form an orthonormal basis of the eigenspace, can be expressed in terms of the eigenvalues of A and its various principal sub-matrices that are obtained by deleting several rows and columns of the same indices up to the order determined by the multiplicity of the eigenvalue.

The question remains whether a similar formula is available when the eigenvalue is not semi-simple. The present paper aims to address this problem. We shall give an eigenvector-eigenvalue identity for the sum of the component moduli of the eigenvector and generalized eigenvectors associated to an eigenvalue that is not semi-simple and corresponds to only one Jordan block in the Jordan canonical form of the matrix, based on the concept of the adjugate of a matrix, the differential analysis of the related determinant functions, and a structural analysis for solving the resulting upper-triangular Toeplitz or more general upper triangular system of equations.

The paper is organized as follows. In the next section we give useful terminologies and preliminary results for the preparation of proving the main theorems in Sections 3 and 4. An eigenvector-eigenvalue-identity will be proved in detail in Section 3 with a simple but rather strong assumption. A more general result will be given in Section 4 with a weakened condition. We conclude in Section 5.

2. Basic concepts and three lemmas

A complex number λ is called an *eigenvalue* of an $n\times n$ complex matrix A if there exists a nonzero n-dimensional complex vector v such that $Av=\lambda v$. The vector v is an *eigenvector* of A associated to eigenvalue λ. A number λ is an eigenvalue of A if and only if it is a zero of the n-th order *characteristic polynomial* $\phi(z)=\det(zI-A)$. The power index of the linear factor $z-\lambda$ in the factorization of $\phi(z)$ in the complex field $\mathbb{C}$ is the *algebraic multiplicity* of λ. The null space $N(A-\lambda I)$ is called the *eigenspace* of λ and its dimension is the *geometric multiplicity* of λ. It is well-known that the geometric multiplicity of λ is less than or equal to its algebraic multiplicity. If they are equal, then the eigenvalue λ is said to be *semi-simple*. In particular, a semi-simple eigenvalue of multiplicity one is called *simple*. A matrix A is diagonalizable if there exists a nonsingular matrix V such that $V^{-1}AV$ is a diagonal matrix. The matrix A is diagonalizable if and only if every eigenvalue is semi-simple.

For an eigenvalue λ, the smallest positive integer r satisfying the equality $N[(A-\lambda I)^r] = N[(A-\lambda I)^{r+1}]$ is called the *index* of λ and is denoted by $\nu(\lambda)$. The null space $N[(A-\lambda I)^{\nu(\lambda)}]$ is referred to as the *generalized eigenspace*, and its nonzero elements that are not eigenvectors are called *generalized eigenvectors* associated to eigenvalue λ. It is well-known that $\nu(\lambda) > 1$ if and only if the algebraic multiplicity of λ is greater than its geometric multiplicity.

Although not every matrix is diagonalizable, any square matrix has its *Jordan canonical form*. Namely there exists a nonsingular matrix V such that $J \equiv V^{-1}AV$ is a block diagonal matrix with Jordan matrices as its diagonal blocks. A $j \times j$ *Jordan matrix* $J_j(\lambda)$ associated to eigenvalue λ is a matrix of the form

$$J_j(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & & \ddots & \lambda & 1 \\ 0 & 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}. \tag{2}$$

The Jordan block $J_j(\lambda)$ is nonsingular when $\lambda \neq 0$, and its inverse is

$$J_j(\lambda)^{-1} = \begin{bmatrix} \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} & \cdots & \cdots & (-1)^{j-1}\lambda^{-j} \\ 0 & \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} & \cdots & (-1)^{j-2}\lambda^{-(j-1)} \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & \ddots & \ddots & -\lambda^{-2} \\ 0 & 0 & \cdots & \cdots & 0 & \lambda^{-1} \end{bmatrix}. \tag{3}$$

The major mathematical concept that will be used to prove the main results in the next two sections is that of the adjugate of a matrix. Given a square matrix $M = [m_{ij}]$, the *cofactor matrix* M_{ij} of its (i,j)-entry m_{ij} is the sub-matrix obtained by deleting the i-th row and the j-th column of M (see [1], p. 114). If the two indices i and j are the same, we simplify the notation M_{ii} to M_i. When the cofactor operation is performed in succession, we write $(\cdots(M_{i_1})\cdots)_{i_r}$ as $M_{i_1,\ldots,i_r}$ for the simplicity of notation.

The definition of the *adjugate* adj M of M is as follows. It is a matrix of the same order as M and its (i,j)-entry is $(-1)^{i+j}\det M_{ji}$. It is well-known that adj $M \cdot M = M \cdot \text{adj}\, M = \det M \cdot I$, so when M is nonsingular,

$$\text{adj}\, M = \det M \cdot M^{-1}. \tag{4}$$

The following lemma is Fact 10.12.8 in the book [1] and its proof can be found in the source reference cited therein.

LEMMA 2.1. *Suppose $M = M(z)$ is a differentiable matrix function from an open domain of $\mathbb{C}$ to $\mathbb{C}^{n\times n}$. Then*

$$\frac{d}{dz}\det M(z) = \sum_{i=1}^{n} \det M_i^d(z),$$

where each $M_i^d(z)$ is a matrix of the same order as $M(z)$, which is obtained via differentiating the entries of the i-th row of $M(z)$.

LEMMA 2.2. *Let $M \in \mathbb{C}^{n\times n}$ and $z \in \mathbb{C}$. Then for a nonnegative integer $t < n$, the t-th derivative of the function $\det(zI - M)$ with respect to z is*

$$\begin{aligned}\det{}^{(t)}(zI-M) &= \sum_{i_t=1}^{n-t+1}\cdots\sum_{i_1=1}^{n}\det(zI-M_{i_1,\dots,i_t})\\ &= \sum_{i_t=1}^{n-t+1}\cdots\sum_{i_1=1}^{n}\prod_{k=1}^{n-t}\left[z-\lambda_k(M_{i_1,\dots,i_t})\right],\end{aligned} \tag{5}$$

where $\lambda_k(M_{i_1,\dots,i_t})$ are the eigenvalues of the $(n-t)\times(n-t)$ matrix $M_{i_1,\dots,i_t}$, counting the algebraic multiplicities.

Proof. By Lemma 2.1, the fact that the derivative of the i-th row of $zI - M$ is the i-th row of I, and the Laplace expansion of a determinant along a row,

$$\det{}'(zI-M) = \sum_{i=1}^{n}\det(zI-M)_i^d = \sum_{i=1}^{n}\det(zI-M_i). \tag{6}$$

So (5) follows by using repeatedly (6) and the fact that the determinant of a matrix is the product of its eigenvalues, counting the algebraic multiplicities. □

We also need some properties of Toeplitz upper-triangular matrices.

LEMMA 2.3. *Let $u_1,\dots,u_k$ and $v_1,\dots,v_k$ be two finite sequences of k vectors, and let $c_1,\dots,c_k$ be k numbers that form a $k\times k$ upper-triangular Toeplitz matrix C. Then the following equality holds:*

$$\begin{aligned}&[v_1\cdots v_k]\begin{bmatrix} c_1 & c_2 & \cdots & c_k\\ 0 & c_1 & \ddots & \vdots\\ \vdots & \vdots & \ddots & c_2\\ 0 & 0 & \cdots & c_1\end{bmatrix}\begin{bmatrix}u_1^H\\ u_2^H\\ \vdots\\ u_k^H\end{bmatrix}\\ &= c_1\sum_{i=1}^{k} v_i u_i^H + c_2\sum_{i=1}^{k-1} v_i u_{i+1}^H + \cdots + c_{k-1}\sum_{i=1}^{2} v_i u_{i+k-2}^H + c_k v_1 u_k^H.\end{aligned} \tag{7}$$

Moreover, if C is nonsingular then

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_k \\ 0 & c_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & c_2 \\ 0 & 0 & \cdots & c_1 \end{bmatrix}^{-1} = \frac{1}{c_1}\begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_k \\ 0 & \beta_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \beta_2 \\ 0 & 0 & \cdots & \beta_1 \end{bmatrix},$$

where the β_t are given recursively by

$$\beta_1 = 1,\ \beta_t = -\alpha_2\beta_{t-1} - \cdots - \alpha_{t-1}\beta_2 - \alpha_t,\ \alpha_t = \frac{c_t}{c_1},\ \ t = 2,\ldots,k. \tag{8}$$

Proof. The proof of (7) and $\beta_1 = 1$ is direct. For the remaining part, it is enough to assume that $c_1 = 1$, and so $\alpha_t = c_t$ for $t = 2,\ldots,k$. Then, since the inverse of a Toeplitz upper-triangular matrix is also Toeplitz upper-triangular, letting the $(1,t)$-entries of $C\cdot C^{-1}$ be 0 gives

$$\beta_t + \alpha_2\beta_{t-1} + \cdots + \alpha_{t-1}\beta_2 + \alpha_t = 0,\ \ t = 2,\ldots,k,$$

from which (8) follows. □

3. An eigenvector-eigenvalue-identity

Let $A = [a_{ij}]$ be an $n\times n$ complex matrix such that one eigenvalue of A, which is denoted by λ throughout the paper, is not semi-simple. Hence, A is not diagonalizable. We list all the eigenvalues of A as

$$\lambda_1(A),\ldots,\lambda_m(A),\lambda_{m+1}(A),\ldots,\lambda_n(A),$$

counting the algebraic multiplicities, where $\lambda_1(A) = \cdots = \lambda_m(A) \equiv \lambda$ and $\lambda_k \neq \lambda$ for $k = m+1,\ldots,n$. We assume that $\lambda_{m+1}(A),\ldots,\lambda_n(A)$ are semi-simple eigenvalues for the convenience of some computations below. Also for the simplicity of analysis, we assume that the Jordan canonical form $J = V^{-1}AV$ of A has only one Jordan block associated to eigenvalue λ, which is the first block in J. Thus, $\nu(\lambda) = m$, and with V partitioned as

$$V = [v_1 \cdots v_m\ v_{m+1} \cdots v_n],$$

where $v_1,\ldots,v_n$ are linearly independent eigenvectors or generalized eigenvectors of A, its first m columns $v_1,\ldots,v_m$ satisfy

$$A[v_1 \cdots v_m] = [v_1 \cdots v_m]J_m(\lambda), \tag{9}$$

where $J_m(\lambda)$ is given by (2).

By (9), the eigenvector v_1 and the generalized eigenvectors $v_2,\ldots,v_m$ associated to eigenvalue λ satisfy the recursive relation

$$\begin{cases} (A-\lambda I)v_1 &= 0, \\ (A-\lambda I)v_2 &= v_1, \\ &\vdots \\ (A-\lambda I)v_m &= v_{m-1}, \end{cases}$$

from which $(A-\lambda I)^m v_m = 0$ and $(A-\lambda I)^{m-1} v_m = v_1 \neq 0$. For any complex number z that is not an eigenvalue of A, since $(zI-A)[v_1 \cdots v_m] = -[v_1 \cdots v_m]J_m(\lambda - z)$, we have

$$(zI-A)^{-1}[v_1 \cdots v_m] = -[v_1 \cdots v_m]J_m(\lambda - z)^{-1}.$$

It follows from the inverse formula (3) that

$$\begin{cases} (zI-A)^{-1}v_1 &= (z-\lambda)^{-1}v_1, \\ (zI-A)^{-1}v_2 &= (z-\lambda)^{-2}v_1 + (z-\lambda)^{-1}v_2, \\ &\vdots \\ (zI-A)^{-1}v_m &= (z-\lambda)^{-m}v_1 + \cdots + (z-\lambda)^{-1}v_m. \end{cases} \tag{10}$$

We further assume that the vectors $v_1, \ldots, v_m$ form an orthonormal basis of the generalized eigenspace $N[(\lambda I - A)^m]$, and such an eigenvector and generalized eigenvectors associated to λ are orthogonal to the eigenvectors $v_{m+1}, \ldots, v_n$ associated to $\lambda_{m+1}(A), \ldots, \lambda_n(A)$, namely

$$v_j^H v_k = 0, \ \ \forall\, j = 1, \ldots, m, \ k = m+1, \ldots, n.$$

Let z be a complex number that is not an eigenvalue of A. Then by (4),

$$\mathrm{adj}(zI-A) = \det(zI-A)(zI-A)^{-1}.$$

Since the determinant of a matrix equals the product of all its eigenvalues counting their algebraic multiplicities, for $s = 1, \ldots, n$,

$$\begin{aligned} \mathrm{adj}(zI-A)v_s &= \prod_{k=1}^{n}[z-\lambda_k(A)] \cdot (zI-A)^{-1}v_s \\ &= g(z)(z-\lambda)^m (zI-A)^{-1}v_s, \end{aligned} \tag{11}$$

where $g(z) = \prod_{k=m+1}^{n}[z-\lambda_k(A)]$. Then (11) and (10) imply that

$$\begin{aligned} \mathrm{adj}(zI-A)v_1 &= g(z)(z-\lambda)^{m-1}v_1, \\ \mathrm{adj}(zI-A)v_2 &= g(z)(z-\lambda)^{m-2}[v_1 + (z-\lambda)v_2], \\ &\vdots \\ \mathrm{adj}(zI-A)v_{m-1} &= g(z)(z-\lambda)\left[v_1 + (z-\lambda)v_2 + \cdots + (z-\lambda)^{m-2}v_{m-1}\right], \\ \mathrm{adj}(zI-A)v_m &= g(z)\left[v_1 + (z-\lambda)v_2 + \cdots + (z-\lambda)^{m-1}v_m\right], \end{aligned} \tag{12}$$

so letting $z \to \lambda$ leads to

$$\begin{cases} \mathrm{adj}(\lambda I-A)v_1 &= 0, \\ &\vdots \\ \mathrm{adj}(\lambda I-A)v_{m-1} &= 0, \\ \mathrm{adj}(\lambda I-A)v_m &= g(\lambda)v_1. \end{cases} \tag{13}$$

On the other hand, for $s = m+1,\dots,n$, the eigenvalues λ_s are assumed to be semi-simple and we have

$$(zI-A)^{-1}v_s = (z-\lambda_s)^{-1}v_s.$$

Hence, by (11),

$$\begin{aligned}\operatorname{adj}(zI-A)v_s &= g(z)(z-\lambda)^m(z-\lambda_s)^{-1}v_s \\ &= \left(\prod_{k=m+1,k\neq s}^{n}[z-\lambda_k(A)]\right)(z-\lambda)^m v_s \end{aligned} \tag{14}$$

and when $z\to\lambda$ we find that

$$\operatorname{adj}(\lambda I-A)v_s = 0,\ \ s = m+1,\dots,n. \tag{15}$$

Recall that $v_m \perp \{v_1,\dots,v_{m-1},v_{m+1},\dots,v_n\}$ by the assumption, and since $\{v_1,\dots,v_n\}$ is a basis of $\mathbb{C}^n$, it follows from (13) and (15) that

$$\operatorname{adj}(\lambda I-A) = g(\lambda)v_1v_m^H. \tag{16}$$

Next, we take derivatives of the functions $\operatorname{adj}(zI-A)v_s$ with respect to z for each s. The m formulas of (12) can be written compactly as

$$\operatorname{adj}(zI-A)v_s = g(z)\sum_{k=1}^{s}(z-\lambda)^{m-s+k-1}v_k,\ \ s=1,\dots,m.$$

Let $h_s(z) = \sum_{k=1}^{s}(z-\lambda)^{m-s+k-1}v_k$. Then for $r=0,\dots,m-1$,

$$h_s^{(r)}(z) = r!\sum_{k=s'}^{s}C_{m-s-r+k-1}^{m-s+k-1}(z-\lambda)^{m-s-r+k-1}v_k,$$

where $s' = \max\{1, s-m+r+1\}$ and $C_j^i = i!/[j!(i-j)!]$ is the combination number. Hence, for $s=1,\dots,m$, the t-th derivative of $\operatorname{adj}(zI-A)v_s$ is

$$\begin{aligned}\operatorname{adj}^{(t)}(zI-A)v_s &= [g(z)h_s(z)]^{(t)} = \sum_{r=0}^{t}C_r^t g^{(t-r)}(z)h_s^{(r)}(z) \\ &= \sum_{r=0}^{t}r!C_r^t g^{(t-r)}(z)\sum_{k=s'}^{s}C_{m-s-r+k-1}^{m-s+k-1}(z-\lambda)^{m-s-r+k-1}v_k. \end{aligned} \tag{17}$$

For any $t=0,1,\dots,m-1$ and $s\leqslant m-t-1$, since $k\geqslant 1$ and $r\leqslant t$ in (17), $m-s-r+k-1 \geqslant m-s-r \geqslant m-s-t \geqslant 1$. Thus, all the powers of $z-\lambda$ in the summation of (17) have positive indices, from which

$$\operatorname{adj}^{(t)}(\lambda I-A)v_s = \lim_{z\to\lambda}\operatorname{adj}^{(t)}(zI-A)v_s = 0,\ \ \forall\, s=1,\dots,m-t-1.$$

On the other hand, for $s = m-t,\ldots,m$, since the powers of $z-\lambda$ in the inner summation of (17) have zero index in the first term and positive indices in the following terms, we see that

$$\begin{aligned}\operatorname{adj}^{(t)}(\lambda I-A)v_s &= \sum_{r=0}^{t} C_r^t g^{(t-r)}(\lambda) r! C_0^r v_{s-m+r+1} \\ &= t!\sum_{r=0}^{t}\frac{g^{(t-r)}(\lambda)}{(t-r)!}v_{s-m+r+1} = t!\sum_{r=m-s}^{t}\frac{g^{(t-r)}(\lambda)}{(t-r)!}v_{s-m+r+1},\end{aligned}$$

where we have used the convention that $v_{s-m+r+1}=0$ when $s-m+r+1<1$.

In particular, as an illustration of the above formula, when $t=1$, we have

$$\operatorname{adj}'(\lambda I-A)v_{m-1} = g(\lambda)v_1 \ \text{ and } \ \operatorname{adj}'(\lambda I-A)v_m = g'(\lambda)v_1 + g(\lambda)v_2, \tag{18}$$

and if $t=m-1$, then for $s=1,\ldots,m$,

$$\operatorname{adj}^{(m-1)}(\lambda I-A)v_s = (m-1)!\left[\frac{g^{(s-1)}(\lambda)}{(s-1)!}v_1+\cdots+\frac{g(\lambda)}{0!}v_s\right].$$

Furthermore, from (14) we obtain that

$$\operatorname{adj}^{(t)}(\lambda I-A)v_s = 0,\ \ \forall\, s=m+1,\ldots,n,\ t=0,\ldots,m-1.$$

We are ready to find the expressions of the derivatives $\operatorname{adj}^{(t)}(\lambda I-A)$ for $t=1,\ldots,m-1$, following the same idea as for the expression (16) of $\operatorname{adj}(\lambda I-A)$. By (18) and since that

$$\operatorname{adj}'(\lambda I-A)v_s = 0,\ s=1,\ldots,m-2,m+1,\ldots,n,$$

$$\operatorname{adj}'(\lambda I-A) = [v_1\ \ v_2]\begin{bmatrix} g(\lambda) & g'(\lambda)\\ 0 & g(\lambda)\end{bmatrix}\begin{bmatrix} v_{m-1}^H\\ v_m^H\end{bmatrix}.$$

Similarly,

$$\operatorname{adj}''(\lambda I-A) = [v_1\ \ v_2\ \ v_3]\begin{bmatrix} 2g(\lambda) & 2g'(\lambda) & g''(\lambda)\\ 0 & 2g(\lambda) & 2g'(\lambda)\\ 0 & 0 & 2g(\lambda)\end{bmatrix}\begin{bmatrix} v_{m-2}^H\\ v_{m-1}^H\\ v_m^H\end{bmatrix}.$$

In general, for $t=0,1,\ldots,m-1$,

$$\operatorname{adj}^{(t)}(\lambda I-A) = t!\,[v_1\cdots v_{t+1}]\begin{bmatrix} g(\lambda) & g'(\lambda) & \cdots & \frac{g^{(t)}(\lambda)}{t!}\\ 0 & g(\lambda) & \ddots & \vdots\\ \vdots & \vdots & \ddots & g'(\lambda)\\ 0 & 0 & \cdots & g(\lambda)\end{bmatrix}\begin{bmatrix} v_{m-t}^H\\ v_{m-t+1}^H\\ \vdots\\ v_m^H\end{bmatrix}.$$

Hence, by (7) of Lemma 2.3, for $t = 0, 1, \ldots, m-1$,

$$\operatorname{adj}^{(t)}(\lambda I - A) = t! \left[\frac{g(\lambda)}{0!} \sum_{i=1}^{t+1} v_i v_{m-t+i-1}^H + \frac{g'(\lambda)}{1!} \sum_{i=1}^{t} v_i v_{m-t+i}^H + \cdots + \frac{g^{(t-1)}(\lambda)}{(t-1)!} \sum_{i=1}^{2} v_i v_{m+i-2}^H + \frac{g^{(t)}(\lambda)}{t!} v_1 v_m^H \right].$$

More explicitly, we have

$$\begin{cases} \operatorname{adj}^{(0)}(\lambda I - A) &= 0! \left[\frac{g(\lambda)}{0!} v_1 v_m^H \right], \\ \operatorname{adj}^{(1)}(\lambda I - A) &= 1! \left[\frac{g(\lambda)}{0!} \left(v_1 v_{m-1}^H + v_2 v_m^H \right) + \frac{g'(\lambda)}{1!} v_1 v_m^H \right], \\ \operatorname{adj}^{(2)}(\lambda I - A) &= 2! \left[\frac{g(\lambda)}{0!} \left(v_1 v_{m-2}^H + v_2 v_{m-1}^H + v_3 v_m^H \right) \right. \\ & \quad \left. + \frac{g'(\lambda)}{1!} \left(v_1 v_{m-1}^H + v_2 v_m^H \right) + \frac{g''(\lambda)}{2!} v_1 v_m^H \right], \\ & \vdots \\ \operatorname{adj}^{(m-1)}(\lambda I - A) &= (m-1)! \left[\frac{g(\lambda)}{0!} \sum_{i=1}^{m} v_i v_i^H + \frac{g'(\lambda)}{1!} \sum_{i=1}^{m-1} v_i v_{i+1}^H + \right. \\ & \quad \left. \cdots + \frac{g^{(m-2)}(\lambda)}{(m-2)!} \sum_{i=1}^{2} v_i v_{m+i-2}^H + \frac{g^{(m-1)}(\lambda)}{(m-1)!} v_1 v_m^H \right]. \end{cases}$$

For $t = 1, \ldots, m$, let

$$B_t = \frac{\operatorname{adj}^{(m-t)}(\lambda I - A)}{(m-t)!} \quad \text{and} \quad X_t = v_1 v_t^H + v_2 v_{t+1}^H + \cdots + v_{m-t+1} v_m^H$$

be the $n \times n$ matrices. Then the above system can be written as

$$\begin{bmatrix} \frac{g(\lambda)}{0!} & \frac{g'(\lambda)}{1!} & \frac{g''(\lambda)}{2!} & \cdots & \frac{g^{(m-1)}(\lambda)}{(m-1)!} \\ 0 & \frac{g(\lambda)}{0!} & \frac{g'(\lambda)}{1!} & \cdots & \frac{g^{(m-2)}(\lambda)}{(m-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{g(\lambda)}{0!} & \frac{g'(\lambda)}{1!} \\ 0 & 0 & \cdots & 0 & \frac{g(\lambda)}{0!} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{m-1} \\ X_m \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{m-1} \\ B_m \end{bmatrix}.$$

From the matrix inverse expression of Lemma 2.3,

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{m-1} \\ X_m \end{bmatrix} = \frac{1}{g(\lambda)} \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_m \\ 0 & \beta_1 & \beta_2 & \cdots & \beta_{m-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_1 & \beta_2 \\ 0 & 0 & \cdots & 0 & \beta_1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{m-1} \\ B_m \end{bmatrix},$$

where β_t are given by (8), in which $\alpha_t = g^{(t-1)}(\lambda)/[(t-1)!g(\lambda)]$ for $t = 2, \ldots, m$. Consequently,

$$\sum_{i=1}^{m} v_i v_i^H = X_1 = \frac{1}{g(\lambda)} \sum_{t=1}^{m} \beta_t B_t = \frac{1}{g(\lambda)} \sum_{t=1}^{m} \beta_t \frac{\operatorname{adj}^{(m-t)}(\lambda I - A)}{(m-t)!}. \tag{19}$$

We are ready to state and prove the main result of this paper.

THEOREM 3.1. *Suppose A is an $n \times n$ complex matrix with eigenvalues*

$$\lambda, \dots, \lambda, \lambda_{m+1}(A), \dots, \lambda_n(A),$$

counting the algebraic multiplicities, such that $\nu(\lambda) = m$ and $\lambda_i(A) \neq \lambda$ for $i = m+1, \dots, n$. Assume that the Jordan form $J = V^{-1}AV$ of A has $J_m(\lambda)$ as the first Jordan block of J and the eigenvalues $\lambda_{m+1}(A), \dots, \lambda_n(A)$ are semi-simple. If the first m columns of $V = [v_1 \cdots v_n]$ form an orthonormal basis of $N[(\lambda I - A)^m]$ and v_j is orthogonal to v_k for all $j = 1, \dots, m$ and $k = m+1, \dots, n$, then for $i = 1, \dots, n$, the following eigenvector-eigenvalue-identity holds for the i-th components v_{ij} of v_j with $j = 1, \dots, m$:

$$\sum_{j=1}^{m} |v_{ij}|^2 = \frac{1}{g(\lambda)} \sum_{t=1}^{m} \frac{\beta_t}{(m-t)!} \sum_{i_{m-t}=1}^{n-m+t} \cdots \sum_{i_1=1}^{n-1} \prod_{k=1}^{n-m+t-1} \left\{\lambda - \lambda_k \left[(A_i)_{i_1,\dots,i_{m-t}}\right]\right\}, \tag{20}$$

where $g(\lambda) = \prod_{k=m+1}^{n} [\lambda - \lambda_k(A)], \beta_1, \dots, \beta_m$ are given by (8), in which $\alpha_t = \frac{g^{(t-1)}(\lambda)}{(t-1)!g(\lambda)}$ for $t = 2, \dots, m$, and $\lambda_k \left[(A_i)_{i_1,\dots,i_{m-t}}\right]$ are the eigenvalues of the matrix $(A_i)_{i_1,\dots,i_{m-t}}$ for $k = 1, \dots, n-m+t-1$ and $t = 1, \dots, m$.

Proof. By the definition of the adjugate, the (i,i)-entry of $\operatorname{adj}^{(m-t)}(\lambda I - A)$ is $\det^{(m-t)}(\lambda I - A_i)$, which is, by (5) of Lemma 2.2,

$$\det{}^{(m-t)}(\lambda I - A_i) = \sum_{i_{m-t}=1}^{n-m+t} \cdots \sum_{i_1=1}^{n-1} \prod_{k=1}^{n-m+t-1} \left\{\lambda - \lambda_k \left[(A_i)_{i_1,\dots,i_{m-t}}\right]\right\}.$$

Here, the summation notations disappear by convention when $t = m$.

Since the (i,i)-entry of the left-hand side of the matrix equality (19) is $\sum_{j=1}^{m} |v_{ij}|^2$, by equating it to the (i,i) entry of the right-hand side of (19), we obtain the desired formula (20) for $\sum_{j=1}^{m} |v_{ij}|^2$. □

REMARK. For the matrix $A = VJV^{-1}$ satisfying the conditions of Theorem 3.1, let $B = Q^H AQ$ and $U = Q^H V$ with Q a unitary matrix. Then $B = UJU^{-1}$, so B has the same Jordan canonical form as A. Since the unitary matrix Q^H preserves the 2-norm and orthogonality of vectors, B also satisfies the conditions of the theorem. However, even though A and B have the same eigenvalues, the equality $\sum_{j=1}^{m} |u_{ij}|^2 = \sum_{j=1}^{m} |v_{ij}|^2$ is not valid in general, because it is not guaranteed that $[u_1 \cdots u_m] = [v_1 \cdots v_m]T$ for some unitary matrix T.

Adding up (20) for $i = 1, \dots, n$ gives the following equality among all the eigenvalues of A and its principal sub-matrices of various orders.

COROLLARY 3.1. *Under the same conditions of Theorem 3.1,*

$$m\prod_{k=m+1}^{n}[\lambda-\lambda_k(A)]$$
$$=\sum_{i=1}^{n}\sum_{t=1}^{m}\frac{\beta_t}{(m-t)!}\sum_{i_{m-t}=1}^{n-m+t}\cdots\sum_{i_1=1}^{n-1}\prod_{k=1}^{n-m+t-1}\left\{\lambda-\lambda_k\left[(A_i)_{i_1,\dots,i_{m-t}}\right]\right\}.$$

To have a taste of Theorem 3.1, we present a special case as a corollary.

COROLLARY 3.2. *Under the same conditions of Theorem 3.1, if* $m=3$, *then*

$$|v_{i1}|^2+|v_{i2}|^2+|v_{i3}|^2$$
$$=\frac{1}{2g(\lambda)}\sum_{i_2=1}^{n-2}\sum_{i_1=1}^{n-1}\prod_{k=1}^{n-3}\{\lambda-\lambda_k[(A_i)_{i_1,i_2}]\}-\frac{g'(\lambda)}{g(\lambda)^2}\sum_{i_1=1}^{n-1}\prod_{k=1}^{n-2}\{\lambda-\lambda_k[(A_i)_{i_1}]\}$$
$$+\left[\frac{g'(\lambda)^2}{g(\lambda)^3}-\frac{g''(\lambda)}{2g(\lambda)^2}\right]\prod_{k=1}^{n-1}[\lambda-\lambda_k(A_i)],\ \ i=1,\dots,n. \tag{21}$$

We illustrate the above identity with the following example.

EXAMPLE. Consider the matrix

$$A=\frac{1}{9}\begin{bmatrix}2&5&-4&0\\2&-4&5&0\\8&2&2&0\\0&0&0&9\end{bmatrix}.$$

It has two distinct eigenvalues 0 and 1 with $\nu(0)=3$. The four vectors

$$v_1=\frac{1}{3}\begin{bmatrix}1\\-2\\-2\\0\end{bmatrix},\ v_2=\frac{1}{3}\begin{bmatrix}-2\\1\\-2\\0\end{bmatrix},\ v_3=\frac{1}{3}\begin{bmatrix}-2\\-2\\1\\0\end{bmatrix},\ v_4=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

constitute an orthonormal basis of $\mathbb{C}^4$ such that

$$Av_1=0,\ Av_2=v_1,\ Av_3=v_2,\ Av_4=v_4,$$

so they satisfy the conditions of Theorem 3.1.

Now, from

$$A_1=\frac{1}{9}\begin{bmatrix}-4&5&0\\2&2&0\\0&0&9\end{bmatrix},$$

we have

$$(A_1)_1=\frac{1}{9}\begin{bmatrix}2&0\\0&9\end{bmatrix},\ (A_1)_2=\frac{1}{9}\begin{bmatrix}-4&0\\0&9\end{bmatrix},\ (A_1)_3=\frac{1}{9}\begin{bmatrix}-4&5\\2&2\end{bmatrix}.$$

It follows that

$$\lambda_1(A_1) = 1,\ \lambda_2(A_1) = \frac{\sqrt{19}-1}{9},\ \lambda_3(A_1) = -\frac{\sqrt{19}+1}{9};$$

$$\lambda_1[(A_1)_1] = \frac{2}{9},\ \lambda_2[(A_1)_1] = 1, \lambda_1[(A_1)_2] = -\frac{4}{9},\ \lambda_2[(A_1)_2] = 1,$$

$$\lambda_1[(A_1)_3] = \frac{\sqrt{19}-1}{9},\ \lambda_2[(A_1)_3] = -\frac{\sqrt{19}+1}{9};$$

$$\lambda_1[(A_1)_{1,1}] = 1,\ \lambda_1[(A_1)_{2,1}] = 1,\ \lambda_1[(A_1)_{3,1}] = \frac{2}{9},$$

$$\lambda_1[(A_1)_{1,2}] = \frac{2}{9},\ \lambda_1[(A_1)_{2,2}] = -\frac{4}{9},\ \lambda_1[(A_1)_{3,2}] = -\frac{4}{9}.$$

Since $g(z) = z-1$, we see that $g(0) = -1, g'(0) = 1$, and $g''(0) = 0$. Thus the right-hand side of (21) with $n = 4$ and $i = 1$ is

$$\begin{aligned}
&\frac{1}{2}\sum_{i_2=1}^{2}\sum_{i_1=1}^{3}\lambda_1\left[(A_1)_{i_1,i_2}\right] - \sum_{i_1=1}^{3}\prod_{k=1}^{2}\lambda_k\left[(A_1)_{i_1}\right] + \prod_{k=1}^{3}\lambda_k(A_1) \\
&= \frac{1}{2}\left(1+1+\frac{2}{9}+\frac{2}{9}-\frac{4}{9}-\frac{4}{9}\right) \\
&\quad -\frac{2}{9}+\frac{4}{9}+\frac{\sqrt{19}-1}{9}\cdot\frac{\sqrt{19}+1}{9}-\frac{\sqrt{19}-1}{9}\cdot\frac{\sqrt{19}+1}{9} \\
&= \frac{7}{9}+\frac{4}{9}-\frac{2}{9} = 1 = |v_{11}|^2+|v_{12}|^2+|v_{13}|^2,
\end{aligned}$$

which is the left-hand side of (21) with $i = 1$. This verifies Corollary 3.2 for $i = 1$. The other identities for the components of $i = 2,3,4$ can be verified in the same way.

4. An extended result

The assumption that $v_1,\ldots,v_m$ satisfying $A[v_1\cdots v_m] = [v_1\cdots v_m]J_m(\lambda)$ form an orthonormal basis of the generalized eigenspace $N((A-\lambda I)^m)$ is quite strong in Theorem 3.1. In this section we try to weaken it to get a similar result. The idea is to change the Jordan block $J_m(\lambda)$ to an upper triangular matrix with diagonal elements λ. This will provide more freedom for the orthogonality condition required in the eigenvector-eigenvalue-identity. For the simplicity of computation and presentation, we only give the generalization for $m = 3$, but the idea is exactly the same for general m.

As in the last section, suppose A has a non-semi-simple eigenvalue λ of algebraic multiplicity 3 and geometric multiplicity 1. Let u_1,u_2,u_3 be an orthonomal basis of $N[(A-\lambda I)^3]$ such that

$$A[u_1\ u_2\ u_3] = [u_1\ u_2\ u_3]\Lambda;\ \ \Lambda = \begin{bmatrix}\lambda & a & b\\ 0 & \lambda & c\\ 0 & 0 & \lambda\end{bmatrix}. \tag{22}$$

The condition (22) implies that $u_j \in N[(A-\lambda I)^j] \setminus N[(A-\lambda I)^{j-1}]$ for $j=1,2,3$. Conversely, if $u_j \in N[(A-\lambda I)^j] \setminus N[(A-\lambda I)^{j-1}]$ for $j=1,2,3$, then $(A-\lambda I)u_1 = 0, (A-\lambda I)u_2 = au_1, (A-\lambda I)u_3 = bu_1 + cu_2$ for some numbers a,b,c, that is (22) is satisfied. Also it is easy to see that $a = u_1^H A u_2, b = u_1^H A u_3$, and $c = u_2^H A u_3$.

We further note that if (9) is satisfied for a basis v_1, v_2, v_3 of $N[(A-\lambda I)^3]$, then there is an orthonomal basis u_1, u_2, u_3 of $N[(A-\lambda I)^3]$ that satisfies (22) for some constants a,b,c. In fact, since the Jordan vectors $v_j \in N[(A-\lambda I)^j] \setminus N[(A-\lambda I)^{j-1}]$ for $j=1,2,3$, starting with $u_1 = v_1/\sqrt{v_1^H v_1}$, the classic Gram-Schmidt orthogonalization process will produce an orthonormal basis u_1,u_2,u_3 of $N[(A-\lambda I)^3]$ that satisfies (22). This shows that there exist orthonormal bases of $N[(A-\lambda I)^3]$ that satisfy the weakened condition.

The condition (22) implies that $(zI-A)[u_1\ u_2\ u_3] = [u_1 \cdots u_3](zI-\Lambda)$ for any complex number z, so when z is not an eigenvalue of A,

$$(zI-A)^{-1}[u_1\ u_2\ u_3] = [u_1\ u_2\ u_3](zI-\Lambda)^{-1},$$

where

$$(zI-\Lambda)^{-1} = (z-\lambda)^{-3}\begin{bmatrix} (z-\lambda)^2 & a(z-\lambda) & b(z-\lambda)+ac \\ 0 & (z-\lambda)^2 & c(z-\lambda) \\ 0 & 0 & (z-\lambda)^2 \end{bmatrix}.$$

Therefore,

$$\begin{cases} (zI-A)^{-1}u_1 = (z-\lambda)^{-1}u_1, \\ (zI-A)^{-1}u_2 = a(z-\lambda)^{-2}u_1 + (z-\lambda)^{-1}u_2, \\ (zI-A)^{-1}u_3 = [b(z-\lambda)^{-2} + ac(z-\lambda)^{-3}]u_1 + c(z-\lambda)^{-2}u_2 \\ \qquad\qquad + (z-\lambda)^{-1}u_3. \end{cases}$$

Since $\operatorname{adj}(zI-A)u_s = g(z)(z-\lambda)^3(zI-A)^{-1}u_s$ for $s=1,2,3$,

$$\begin{cases} \operatorname{adj}(zI-A)u_1 = g(z)(z-\lambda)^2u_1, \\ \operatorname{adj}(zI-A)u_2 = g(z)[a(z-\lambda)u_1 + (z-\lambda)^2u_2], \\ \operatorname{adj}(zI-A)u_3 = g(z)\{[b(z-\lambda)+ac]u_1 + c(z-\lambda)u_2 \\ \qquad\qquad + (z-\lambda)^2u_3\}. \end{cases} \tag{23}$$

Letting $z \to \lambda$, we have

$$\begin{cases} \operatorname{adj}(\lambda I-A)u_1 = 0, \\ \operatorname{adj}(\lambda I-A)u_2 = 0, \\ \operatorname{adj}(\lambda I-A)u_3 = acg(\lambda)u_1. \end{cases}$$

It follows from the above and (15) that

$$\operatorname{adj}(\lambda I-A) = acg(\lambda)u_1u_3^H. \tag{24}$$

Differentiating the expression of each $\operatorname{adj}(zI-A)u_s$ in (23) gives that

$$\begin{cases} \operatorname{adj}'(zI-A)u_1 = [g'(z)(z-\lambda)^2 + 2g(z)(z-\lambda)]u_1, \\ \operatorname{adj}''(zI-A)u_1 = [g''(z)(z-\lambda)^2 + 4g'(z)(z-\lambda) + 2g(z)]u_1; \end{cases}$$

$$\begin{cases} \mathrm{adj}'(zI-A)u_2 &= a[g'(z)(z-\lambda)+g(z)]u_1 \\ &\quad +[g'(z)(z-\lambda)^2+2g(z)(z-\lambda)]u_2, \\ \mathrm{adj}''(zI-A)u_2 &= a[g''(z)(z-\lambda)+2g'(z)]u_1 \\ &\quad +[g''(z)(z-\lambda)^2+4g'(z)(z-\lambda)+2g(z)]u_2; \end{cases}$$

$$\begin{cases} \mathrm{adj}'(zI-A)u_3 &= [bg'(z)(z-\lambda)+acg'(z)+bg(z)]u_1 \\ &\quad +c[g'(z)(z-\lambda)+g(z)]u_2 \\ &\quad +[g'(z)(z-\lambda)^2+2g(z)(z-\lambda)]u_3, \\ \mathrm{adj}''(zI-A)u_3 &= [bg''(z)(z-\lambda)+acg''(z)+2bg'(z)]u_1 \\ &\quad +c[g''(z)(z-\lambda)+2g'(z)]u_2 \\ &\quad +[g''(z)(z-\lambda)^2+4g'(z)(z-\lambda)+2g(z)]u_3. \end{cases}$$

Taking the limit of $z \to \lambda$, we see that

$$\begin{cases} \mathrm{adj}'(\lambda I-A)u_1 &= 0, \\ \mathrm{adj}'(\lambda I-A)u_2 &= ag(\lambda)u_1, \\ \mathrm{adj}'(\lambda I-A)u_3 &= [acg'(\lambda)+bg(\lambda)]u_1+cg(\lambda)u_2, \\ \mathrm{adj}''(\lambda I-A)u_1 &= 2g(\lambda)u_1, \\ \mathrm{adj}''(\lambda I-A)u_2 &= 2ag'(\lambda)u_1+2g(\lambda)u_2, \\ \mathrm{adj}''(\lambda I-A)u_3 &= [acg''(\lambda)+2bg'(\lambda)]u_1+2cg'(\lambda)u_2+2g(\lambda)u_3. \end{cases}$$

By the same method as in the previous section, we find that

$$\mathrm{adj}'(\lambda I-A) = [u_1\ u_2]\begin{bmatrix} ag(\lambda) & acg'(\lambda)+bg(\lambda) \\ 0 & cg(\lambda) \end{bmatrix}\begin{bmatrix} u_2^H \\ u_3^H \end{bmatrix},$$

$$\mathrm{adj}''(\lambda I-A) = [u_1\ u_2\ u_3]\begin{bmatrix} 2g(\lambda) & 2ag'(\lambda) & acg''(\lambda)+2bg'(\lambda) \\ 0 & 2g(\lambda) & 2cg'(\lambda) \\ 0 & 0 & 2g(\lambda) \end{bmatrix}\begin{bmatrix} u_1^H \\ u_2^H \\ u_3^H \end{bmatrix}.$$

Together with (24), we have

$$\begin{aligned} \mathrm{adj}(\lambda I-A) &= acg(\lambda)u_1u_3^H, \\ \mathrm{adj}'(\lambda I-A) &= g(\lambda)(au_1u_2^H+cu_2u_3^H)+[acg'(\lambda)+bg(\lambda)]u_1u_3^H, \\ \mathrm{adj}''(\lambda I-A) &= 2g(\lambda)(u_1u_1^H+u_2u_2^H+u_3u_3^H)+2g'(\lambda)(au_1u_2^H+cu_2u_3^H) \\ &\quad +[acg''(\lambda)+2bg'(\lambda)]u_1u_3^H, \end{aligned}$$

which can be written as

$$\begin{bmatrix} g(\lambda) & g'(\lambda) & \frac{ac}{2}g''(\lambda)+bg(\lambda) \\ 0 & g(\lambda) & acg'(\lambda)+bg(\lambda) \\ 0 & 0 & acg(\lambda) \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{adj}''(\lambda I-A)}{2} \\ \mathrm{adj}'(\lambda I-A) \\ \mathrm{adj}(\lambda I-A) \end{bmatrix},$$

where $X_1 = u_1u_1^H + u_2u_2^H + u_3u_3^H, X_2 = au_1u_2^H + cu_2u_3^H, X_3 = u_1u_3^H$. Thus

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} g(\lambda) & g'(\lambda) & \frac{ac}{2}g''(\lambda)+bg(\lambda) \\ 0 & g(\lambda) & acg'(\lambda)+bg(\lambda) \\ 0 & 0 & acg(\lambda) \end{bmatrix}^{-1} \begin{bmatrix} \frac{\mathrm{adj}''(\lambda I-A)}{2} \\ \mathrm{adj}'(\lambda I-A) \\ \mathrm{adj}(\lambda I-A) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{g(\lambda)} & -\frac{g'(\lambda)}{g(\lambda)^2} & \frac{g'(\lambda)^2}{g(\lambda)^3} - \frac{g''(\lambda)}{2g(\lambda)^2} + \frac{b}{ac}\left(\frac{g'(\lambda)}{g(\lambda)^2} - \frac{1}{g(\lambda)}\right) \\ 0 & \frac{1}{g(\lambda)} & -\frac{g'(\lambda)}{g(\lambda)^2} - \frac{b}{acg(\lambda)} \\ 0 & 0 & \frac{1}{acg(\lambda)} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{adj}''(\lambda I-A)}{2} \\ \mathrm{adj}'(\lambda I-A) \\ \mathrm{adj}(\lambda I-A) \end{bmatrix}.$$

Consequently,

$$u_1u_1^H + u_2u_2^H + u_3u_3^H = \frac{\mathrm{adj}''(\lambda I-A)}{2g(\lambda)} - \frac{g'(\lambda)}{g(\lambda)^2}\mathrm{adj}'(\lambda I-A)$$
$$+ \left[\frac{g'(\lambda)^2}{g(\lambda)^3} - \frac{g''(\lambda)}{2g(\lambda)^2} + \frac{b}{ac}\left(\frac{g'(\lambda)}{g(\lambda)^2} - \frac{1}{g(\lambda)}\right)\right]\mathrm{adj}(\lambda I-A).$$

THEOREM 4.1. *Suppose A is an $n\times n$ complex matrix with eigenvalues*

$$\lambda, \lambda, \lambda, \lambda_4(A), \ldots, \lambda_n(A),$$

counting the algebraic multiplicities, such that $\nu(\lambda) = 3$ and $\lambda_i \neq \lambda$ for $i = 4, \ldots, n$. Assume that the eigenvalues $\lambda_4(A), \ldots, \lambda_n(A)$ are semi-simple. If u_1, u_2, u_3 form an orthonormal basis of $N[(\lambda I - A)^3]$ that satisfies (22) and they are orthogonal to each of linearly independent eigenvectors $u_4, \ldots, u_n$ of A associated to the eigenvalues $\lambda_4(A)$, $\ldots, \lambda_n(A)$ respectively, then for $i = 1, \ldots, n$, the following eigenvector-eigenvalue-identity holds for the i-th components u_{ij} of u_j with $j = 1, 2, 3$:

$$|u_{i1}|^2 + |u_{i2}|^2 + |u_{i3}|^2$$
$$= \frac{1}{2g(\lambda)} \sum_{i_2=1}^{n-2} \sum_{i_1=1}^{n-1} \prod_{k=1}^{n-3} \{\lambda - \lambda_k[(A_i)_{i_1,i_2}]\} - \frac{g'(\lambda)}{g(\lambda)^2} \sum_{i_1=1}^{n-1} \prod_{k=1}^{n-2} \{\lambda - \lambda_k[(A_i)_{i_1}]\}$$
$$+ \left[\frac{g'(\lambda)^2}{g(\lambda)^3} - \frac{g''(\lambda)}{2g(\lambda)^2} + \frac{b}{ac}\left(\frac{g'(\lambda)}{g(\lambda)^2} - \frac{1}{g(\lambda)}\right)\right] \prod_{k=1}^{n-1} [\lambda - \lambda_k(A_i)],$$

where $g(\lambda) = \prod_{k=4}^{n}[\lambda - \lambda_k(A)]$.

REMARK. When $a = c = 1$ and $b = 0$, Theorem 4.1 is reduced to Theorem 3.1 with $m = 3$.

5. Conclusions

We have obtained a new eigenvector-eigenvalue-identity for a general matrix such that one eigenvalue corresponds to exactly one Jordan block of order more than one, under a proper orthogonality condition among the eigenvectors and generalized eigenvectors that form a basis of the underlying unitary space. This supplements the eigenvector-eigenvalue-identity when the eigenvalue is semi-simple with multiplicity more than one. Such results provide a basis for the eigenvector-eigenvalue-identity of an arbitrary matrix, that is, the eigenvalue may have several Jordan blocks in the canonical form. The study of the most general case will be done in the future.

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