

GENERALIZED NUMERICAL RADIUS AND RELATED INEQUALITIES

T. BOTTAZZI* AND C. CONDE

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Abstract. In [2], Abu Omar and Kittaneh defined a new generalization of the numerical radius. That is, given a norm $N(\cdot)$ on $\mathcal{B}(H)$, the space of bounded linear operators over a Hilbert space H , and $A \in \mathcal{B}(H)$

$$w_N(A) = \sup_{\theta \in \mathbb{R}} N(\operatorname{Re}(e^{i\theta} A)).$$

They proved several properties and introduced some inequalities. We continue with the study of this generalized numerical radius and we develop diverse inequalities involving w_N . We also study particular cases when $N(\cdot)$ is the p -Schatten norm with $p > 1$.

1. Introduction

Throughout the paper, $(H, \langle \cdot, \cdot \rangle)$ denotes a separable complex Hilbert space. Let $\mathcal{B}(H)$ and $\mathcal{K}(H)$ denote the Banach spaces of all bounded operators and all compact operators equipped with the usual operator norm $\|\cdot\|$, respectively. In the case when $\dim(H) = n$, we identify $\mathcal{B}(H)$ with the full matrix algebra \mathbb{M}_n of all $n \times n$ matrices with entries in the complex field. The symbol I stands for the identity operator on H .

For $A \in \mathcal{B}(H)$, A^* denotes the adjoint of A and if $A = A^*$ then A is a selfadjoint operator. We can write $A = \operatorname{Re}(A) + i\operatorname{Im}(A)$, in which $\operatorname{Re}(A) = \frac{A+A^*}{2}$ and $\operatorname{Im}(A) = \frac{A-A^*}{2i}$ are selfadjoint operators. This is the so called Cartesian decomposition of A .

For each operator $A \in \mathcal{B}(H)$, we consider the numerical range $W(A) = \{\langle Ah, h \rangle : \|h\| = 1\}$ and

$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\},$$

called as numerical radius of A . It is well known that $W(A)$ is a bounded convex subset of the complex plane, which contains the spectrum of A in its closure. The numerical radius $w(\cdot)$ defines a norm on $\mathcal{B}(H)$ which is equivalent to $\|\cdot\|$. In fact, the following inequalities

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|, \tag{1.1}$$

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* Corresponding author.

hold. The first inequality becomes an equality if $A^2 = 0$ and the second inequality becomes an equality if A is normal. On the other hand, Yamazaki proved in [33] that

$$w(A) = \sup_{\theta \in \mathbb{R}} \|Re(e^{i\theta} A)\|.$$

Recently, motivated by the previous relation, the authors in [2] gave a generalization of the numerical radius, in this way: for any $A \in \mathcal{B}(H)$

$$w_N(A) = \sup_{\theta \in \mathbb{R}} N(Re(e^{i\theta} A)), \tag{1.2}$$

where $N(\cdot)$ is a norm on $\mathcal{B}(H)$. We say that $N(\cdot)$ is selfadjoint if $N(A) = N(A^*)$ for any $A \in \mathcal{B}(H)$ and submultiplicative if $N(AB) \leq N(A)N(B)$ for every $A, B \in \mathcal{B}(H)$.

We briefly describe the contents of this paper. Section 2 and 3 contain basic definitions, notation and some preliminary results about Schatten ideals, orthogonality and norm-parallelism of bounded linear operators. In section 4 we obtain new upper and lower bounds for w_N . Section 5 continues in a similar way considering $w_N = w_p$ for p -Schatten norm with $1 \leq p < \infty$ and we focus on characterize the attainment of the upper bound for w_p . Finally, in the last section, we obtain several inequalities involving the norms $N(\cdot)$ and w_N for the product of two operators in $\mathcal{B}(H)$.

2. Some facts on p -Schatten class

For an operator $A \in \mathcal{B}(H)$, we denote the modulus by $|A| = (A^*A)^{\frac{1}{2}}$. For any compact operator $A \in \mathcal{K}(H)$, let $s_1(A), s_2(A), \dots$ be the singular values of A , i.e. the eigenvalues of the $|A|$ in decreasing order and repeated according to multiplicity. For $p > 0$, let

$$\|A\|_p = \left(\sum_{i=1}^{\infty} s_i(A)^p \right)^{\frac{1}{p}} = (\text{tr}|A|^p)^{\frac{1}{p}}, \tag{2.1}$$

where $\text{tr}(\cdot)$ is the trace functional, i.e.

$$\text{tr}(A) = \sum_{j=1}^{\infty} \langle Ae_j, e_j \rangle, \tag{2.2}$$

with $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of H . Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.2) converges absolutely and it is independent from the choice of basis. Equality (2.1) defines a norm (quasi-norm) on the ideal $\mathcal{B}_p(H) = \{A \in \mathcal{K}(H) : \|A\|_p < \infty\}$ for $1 \leq p < \infty$ ($0 < p < 1$), called the p -Schatten class.

The following theorem collects some of the most important properties of p -Schatten operators:

THEOREM 1. 1. $\mathcal{B}_p(H) \subseteq \mathcal{K}(H)$.

2. $\mathcal{B}_0(H)$, the space of operators of finite rank, is a dense subspace of $\mathcal{B}_p(H)$.

3. $\mathcal{B}_p(H)$ is an operator ideal in $\mathcal{B}(H)$ for $1 \leq p < \infty$.
4. If $p_1 < p_2$ and $A \in \mathcal{B}_{p_1}(H)$, then $A \in \mathcal{B}_{p_2}(H)$ and $\|A\|_{p_2} \leq \|A\|_{p_1}$.
5. For any $A \in \mathcal{B}_p(H)$ and $T \in \mathcal{B}(H)$ we have the following inequalities:

$$\|A\| \leq \|A\|_p, \|A\|_p = \|A^*\|_p \text{ and } \|TA\|_p \leq \|T\| \|A\|_p.$$

6. For $p > 0$, $A \in \mathcal{B}_p(H)$ if and only if $A^*A \in \mathcal{B}_{p/2}(H)$ and in this case $\|A\|_p^2 = \|A^*A\|_{p/2}$.

It is known that for $0 < p < 1$, instead of the triangle inequality, which does not hold in this case, we have $\|A + B\|_p^p \leq \|A\|_p^p + \|B\|_p^p$ for $A, B \in \mathcal{B}_p(H)$. The so-called *Hilbert-Schmidt class* $\mathcal{B}_2(H)$ is a Hilbert space under the inner product $\langle A, B \rangle_{HS} := \text{tr}(B^*A)$. The ideal $\mathcal{B}_1(H)$ is called the *trace class*. For $1 < p < \infty$, $(\mathcal{B}_p(H), \|\cdot\|_p)$ is a uniformly convex space as consequence of the classical McCarthy-Clarkson inequality (see [27], Th. 2.7).

If $x, y \in H$, then we denote $x \otimes y$ the rank 1 operator defined on H by $(x \otimes y)(z) = \langle z, y \rangle x$, then $\|x \otimes y\| = \|x\| \|y\| = \|x \otimes y\|_p$.

The usual operator norm and the Schatten p -norms are special examples of unitarily invariant norms, i.e. that satisfies the invariance property $\|UXV\| = \|X\|$, for any pair of unitary operators U, V . On the theory of norm ideals and their associated unitarily invariant norms, a reference for this subject is [16].

3. Orthogonality and norm-parallelism of operators

Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We say that $x \in X$ is *norm parallel* to $y \in X$ ([31]), in short $x \parallel y$, if there exists $\lambda \in \mathbb{T} = \{\alpha \in \mathbb{K} : |\alpha| = 1\}$ such that

$$\|x + \lambda y\| = \|x\| + \|y\|. \tag{3.1}$$

In the framework of inner product spaces or uniformly convex spaces, the norm parallel relation is exactly the usual vectorial parallel relation, that is, $x \parallel y$ if and only if x and y are linearly dependent. In the setting of normed linear spaces, two linearly dependent vectors are norm parallel, but the converse is false in general. Several characterizations of the norm parallelism for Hilbert space operators were given in [17, 34, 35].

The orthogonality between two vectors of X , may be defined in several ways. The so-called *Birkhoff-James* orthogonality reads as follows (see [12, 18]): for $x, y \in X$ it is said that x is *Birkhoff-James orthogonal* (B-J) to y , denoted by $x \perp y$, whenever

$$\|x\| \leq \|x + \gamma y\| \tag{3.2}$$

for all $\gamma \in \mathbb{K}$. If X is an inner product space, then B-J orthogonality is equivalent to the usual orthogonality given by the inner product. It is also easy to see that B-J orthogonality is nondegenerate, is homogeneous, but it is neither symmetric nor additive.

There are other definitions of orthogonality with different properties. Orthogonality in the setting of Hilbert space operators has attracted attention of several mathematicians. We cite some papers which studied these notions in chronological order, see for instance [32, 21, 26, 8, 4, 11, 29, 30, 13].

Let $\mathcal{B}_p(H)$ be a p -Schatten ideal with $p > 0$. According to (3.1), we say that $A, B \in \mathcal{B}_p(H)$ are norm parallel, denoted by $A \parallel^p B$, if there exists $\lambda \in \mathbb{T}$ such that $\|A + \lambda B\|_p = \|A\|_p + \|B\|_p$.

In [13] we characterized the norm-parallelism between two operators in $\mathcal{B}_p(H)$, as follows.

THEOREM 2. *Let $A, B \in \mathcal{B}_p(H)$ with polar decompositions $A = U|A|$ and $B = V|B|$, respectively. If $1 < p < \infty$, then the following conditions are equivalent:*

- (i) $A \parallel^p B$.
- (ii) $\|A\|_p |\operatorname{tr}(|A|^{p-1} U^* B)| = \|B\|_p \operatorname{tr}(|A|^p)$.
- (iii) $\|B\|_p |\operatorname{tr}(|B|^{p-1} V^* A)| = \|A\|_p \operatorname{tr}(|B|^p)$.
- (iv) A, B are linearly dependent.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then, it is trivial that $\|A + I\|_1 = 3 = \|A\|_1 + \|I\|_1$, $\|A + I\| = 2 = \|A\| + \|I\|$. However, it is evident that A and I are linearly independent.

In the last years, different authors have obtained characterizations of the norm parallelism problem for trace-class operators on a Hilbert space H . In the context of an infinite dimensional Hilbert space we refer [24] and [34] and in other hand, for a finite dimensional space we mention [25].

4. Properties of w_N norms

We start this section with basic properties of the norm $w_N(\cdot)$.

PROPOSITION 1. *For any $A \in \mathcal{B}(H)$ holds $w_N(A) = \max_{\theta \in [0, 2\pi]} N(\operatorname{Re}(e^{i\theta} A))$.*

Proof. It is a simple consequence of the following facts: the interval $[0, 2\pi]$ is a compact set and the function $\theta \rightarrow N(\operatorname{Re}(e^{i\theta} A))$ is continuous. \square

LEMMA 1. *For any $A, X \in \mathcal{B}(H)$ the following statements hold.*

1. $w_N(A) = \sup_{\theta \in \mathbb{R}} N(\operatorname{Im}(e^{i\theta} A))$.
2. $w_N(A) = \sup_{\alpha^2 + \beta^2 = 1} N(\alpha \operatorname{Re}(A) + \beta \operatorname{Im}(A))$.

3. $\max\{\frac{1}{2}N(A), \frac{1}{2}N(A^*)\} \leq w_N(A) \leq \frac{1}{2}(N(A) + N(A^*)).$
4. *If $N(\cdot)$ is submultiplicative then $w_N(AX \pm XA^*) \leq (N(A) + N(A^*))w_N(X).$*
5. *If $w_N(A) = \frac{1}{2}N(A)$, then $Re(e^{i\theta}A) \parallel^N Im(e^{i\theta}A)$ for all $\theta \in \mathbb{R}.$*

Proof. Item (4) are proved in [2]. The proofs of the rest are straightforward. \square

We note that the upper bound obtained in item (3) is a simple consequence of the following inequality which holds for any norm N ,

$$N\left(\frac{e^{i\theta}A + e^{-i\theta}A^*}{2}\right) \leq \frac{N(A) + N(A^*)}{2}.$$

This concept is called midpoint point convexity an it was introduced by Jensen. In the context of continuity, midpoint convexity means convexity, and for this reason we have the following inequality

$$N\left(\frac{e^{i\theta}A + e^{-i\theta}A^*}{2}\right) \leq \int_0^1 N((1 - \lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*) d\lambda \leq \frac{N(A) + N(A^*)}{2},$$

as a particular case of the classical Hermite-Hadamard (H-H) inequality for the function $f_\theta(\lambda) = N((1 - \lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*)$ where $\theta \in \mathbb{R}$ and $A \in \mathcal{B}(H)$. For sake of completeness, we recall a refinement of H-H inequality: let f be a real valued function which is convex on the interval $[a, b]$. Then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{4.1}$$

Inspired by the previous result, we derive several inequalities for the generalized numerical radius. In particular we obtain an improvement of (3) in Lemma 1.

THEOREM 3. *Let $A \in \mathcal{B}(H)$ and $N(\cdot)$ be a norm on $\mathcal{B}(H)$. Then*

$$\begin{aligned} w_N(A) &\leq \sup_{\theta \in \mathbb{R}} \int_0^1 N((1 - \lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*) d\lambda \leq \frac{1}{2}w_N(A) + \frac{1}{4}(N(A) + N(A^*)) \\ &\leq \frac{1}{2}(N(A) + N(A^*)). \end{aligned} \tag{4.2}$$

Proof. For each $\theta \in \mathbb{R}$ we consider the function f_θ defined above. It is easy to see that f_θ is a convex function in $[0, 1]$ and so by the previous statement we have

$$\begin{aligned} \frac{1}{2}N(e^{i\theta}A + e^{-i\theta}A^*) &\leq \int_0^1 N((1 - \lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*) d\lambda \\ &\leq \frac{1}{2} \left[\frac{1}{2}N(e^{i\theta}A + e^{-i\theta}A^*) + \frac{1}{2}(N(A) + N(A^*)) \right] \\ &\leq \frac{1}{2}(N(A) + N(A^*)). \end{aligned}$$

Taking the supremum over $\theta \in \mathbb{R}$ we conclude that

$$\begin{aligned} w_N(A) &\leq \sup_{\theta \in \mathbb{R}} \int_0^1 N((1-\lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*) d\lambda \leq \frac{1}{2}w_N(A) + \frac{1}{4}(N(A) + N(A^*)) \\ &\leq \frac{1}{2}(N(A) + N(A^*)). \end{aligned}$$

This completes the proof of the theorem. \square

PROPOSITION 2. *Let $\phi : [0, \infty) \rightarrow \mathbb{R}$ be any nondecreasing convex function or midpoint convex function and $N(\cdot)$ be a norm on $\mathcal{B}(H)$. Then*

$$\begin{aligned} \phi(w_N(A)) &\leq \phi\left(\frac{1}{2}\left[w_N(A) + \frac{1}{2}(N(A) + N(A^*))\right]\right) \\ &\leq \frac{1}{2}\phi(w_N(A)) + \frac{1}{2}\phi\left(\frac{1}{2}(N(A) + N(A^*))\right). \end{aligned}$$

Clearly convexity implies midpoint-convexity. However, there exist midpoint-convex functions that are not convex. Such functions can be very strange and interesting. In particular if $\phi(x) = x^r$ with $r \geq 1$ we have

$$w_N^r(A) \leq \frac{1}{2^r} \left[w_N(A) + \frac{1}{2}(N(A) + N(A^*)) \right]^r \leq \frac{1}{2} w_N^r(A) + \frac{1}{2^{r+1}} (N(A) + N(A^*))^r.$$

REMARK 1. Let $N(\cdot)$ be a norm on $\mathcal{B}(H)$. Following Kikianty and Dragomir in [19] we introduce a norm on $\mathcal{B}(H) \times \mathcal{B}(H)$, which we call the r -HH-norm induced by $N(\cdot)$, and is defined as follows:

$$N((A, B))_{r-HH} := \sup_{\theta \in \mathbb{R}} \left(\int_0^1 N((1-\lambda)e^{i\theta}A + \lambda e^{-i\theta}B^*)^r d\lambda \right)^{\frac{1}{r}},$$

for any $1 \leq r < \infty$ and $(A, B) \in \mathcal{B}(H) \times \mathcal{B}(H)$. From the classical H-H inequality, we obtain

$$\begin{aligned} \sup_{\theta \in \mathbb{R}} N\left(\frac{e^{i\theta}A + e^{-i\theta}B^*}{2}\right) &\leq \sup_{\theta \in \mathbb{R}} \left(\int_0^1 N((1-\lambda)e^{i\theta}A + \lambda e^{-i\theta}B^*)^r d\lambda \right)^{\frac{1}{r}} \\ &\leq \left(\frac{N(A)^r + N(B)^r}{2} \right)^{\frac{1}{r}}, \end{aligned}$$

and in particular if we consider the pair (A, A^*) we get a norm on $\mathcal{B}(H)$ given by $\bar{N}(A)_{r-HH} := N((A, A^*))_{r-HH}$ and

$$w_N(A) \leq \sup_{\theta \in \mathbb{R}} \left(\int_0^1 N((1-\lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*)^r d\lambda \right)^{\frac{1}{r}} \leq \left(\frac{N(A)^r + N(A^*)^r}{2} \right)^{\frac{1}{r}}. \tag{4.3}$$

It is clear that $\overline{N}(\cdot)_{r-HH}$ is a selfadjoint norm on $\mathcal{B}(H)$ and $\overline{N}(A)_{r-HH} = N(A)$ if A is a selfadjoint operator. We note that $\overline{N}(\cdot)_{1-HH}$ and $w_N(\cdot)$ norms are equivalent in $\mathcal{B}(H)$, since by inequality (4.2) we have

$$w_N(A) \leq \sup_{\theta \in \mathbb{R}} \int_0^1 N((1-\lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*) d\lambda \leq \frac{3}{2}w_N(A).$$

PROPOSITION 3. Let $A \in \mathcal{B}(H)$ and $r \geq 1$. Then $w_{\overline{N}_{r-HH}}(A) = w_N(A)$.

Proof. For each $\theta \in \mathbb{R}$ the operator $Re(e^{i\theta}A)$ is selfadjoint, then

$$w_{\overline{N}_{r-HH}}(A) = \sup_{\theta \in \mathbb{R}} \overline{N}_{r-HH}(Re(e^{i\theta}A)) = \sup_{\theta \in \mathbb{R}} N(Re(e^{i\theta}A)) = w_N(A). \quad \square$$

In the next theorem, we establish a lower bound for the generalized numerical radius.

THEOREM 4. Let $A \in \mathcal{B}(H)$ and $N(\cdot)$ be a norm on $\mathcal{B}(H)$. Then

$$0 \leq 2 \sup_{\theta \in \mathbb{R}} \int_0^1 N((1-\lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*) d\lambda - \frac{1}{2}(N(A) + N(A^*)) \leq w_N(A). \quad (4.4)$$

Proof. We have by inequality (4.2)

$$\begin{aligned} 0 &\leq \max \left\{ \frac{1}{2}N(A), \frac{1}{2}N(A^*) \right\} - \frac{1}{4}(N(A) + N(A^*)) \leq w_N(A) - \frac{1}{4}(N(A) + N(A^*)) \\ &\leq \sup_{\theta \in \mathbb{R}} \int_0^1 N((1-\lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*) d\lambda - \frac{1}{4}(N(A) + N(A^*)) \leq \frac{1}{2}w_N(A). \quad \square \end{aligned}$$

COROLLARY 1. Let $A \in \mathcal{B}(H)$ and $N(\cdot)$ be a norm on $\mathcal{B}(H)$. Then

$$\frac{1}{4}(N(A) + N(A^*)) \leq \sup_{\theta \in \mathbb{R}} \int_0^1 N((1-\lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*) d\lambda \leq \frac{1}{2}(N(A) + N(A^*)). \quad (4.5)$$

Proof. It is an immediate consequence of (4.2) and (4.4). \square

Our next goal is to determine when w_N coincides with the upper bound $\frac{1}{2}(N(A) + N(A^*))$. In the context of bounded linear operators on Hilbert spaces, we notice that if $A \in \mathcal{B}(H)$ satisfies $\|A\| = w(A)$ (i.e. A is a normaloid operator) if and only if $A \parallel I$ (see Proposition 4.2 in [13]). In [35], the authors investigated the case when an operator is parallel to the identity operator in the context of $\mathcal{B}(H)$. Combining the previous results we have the following characterization: let $A \in \mathcal{B}(H)$ then

$$w(A) = \|A\| \Leftrightarrow A \parallel I \Leftrightarrow A \parallel A^*.$$

We obtain an analogous result for w_N .

THEOREM 5. *Let $A \in \mathcal{B}(H)$. Then the following conditions are equivalent:*

1. $w_N(A) = \frac{1}{2}(N(A) + N(A^*))$.
2. $A \parallel^N A^*$.

Moreover, if $N(\cdot)$ is selfadjoint the previous conditions are equivalent to $w_N(A) = N(A)$.

Proof. Suppose that $A \parallel^N A^*$, then there exists $\lambda = e^{i2\theta_0} \in \mathbb{T}$ such that $N(A + \lambda A^*) = N(A) + N(A^*)$. This equality implies

$$N\left(\frac{e^{-i\theta_0}A + e^{i\theta_0}A^*}{2}\right) = \frac{1}{2}(N(A) + N(A^*)).$$

On the other hand, if $w_N(A) = \frac{1}{2}(N(A) + N(A^*))$ there exists $\theta_0 \in [0, 2\pi]$ such that $N(\operatorname{Re}(e^{i\theta_0}A)) = \frac{1}{2}(N(A) + N(A^*))$, that is $A \parallel^N A^*$. \square

COROLLARY 2. *Let $A \in \mathcal{B}(H)$ and $N(\cdot)$ a selfadjoint and strictly convex norm on $\mathcal{B}(H)$. Then the following conditions are equivalent:*

1. $w_N(A) = N(A)$.
2. $A \parallel^N A^*$.
3. $A = \alpha A^*$ with $\alpha \in \mathbb{T}$.

5. The case w_p

In this section we focus on the particular case $N(\cdot) = \|\cdot\|_p$ for $1 \leq p < \infty$, that is

$$w_p(A) = \sup_{\theta \in \mathbb{R}} \|\operatorname{Re}(e^{i\theta}A)\|_p,$$

for any $A \in \mathcal{B}_p(H)$.

First we will focus on remembering and obtaining new bounds for $w_2(\cdot)$, for the particular case of the Hilbert-Schmidt norm. In [2], the authors proved an explicit formula of $w_2(\cdot)$ in terms of $\|A\|_2$ and $\operatorname{tr}(A^2)$, more precisely

$$w_2(A) = \sqrt{\frac{\|A\|_2^2 + |\operatorname{tr}(A^2)|}{2}}$$

for any $A \in \mathcal{B}_2(H)$ and the existence of lower and upper bounds, which are

$$\frac{1}{\sqrt{2}}\|A\|_2 \leq w_2(A) \leq \|A\|_2.$$

The following result improve the existence of a new lower bound for $w_2(\cdot)$.

THEOREM 6. Let $A \in \mathcal{B}_2(H)$. Then

$$\max \left\{ \frac{1}{\sqrt{2}} \|A\|_2, \frac{\|A\|_2 + |tr(A^2)|^{1/2}}{2} \right\} \leq w_2(A). \tag{5.1}$$

Proof. For $a, b \geq 0$, the arithmetic mean $\mathcal{A}(a, b)$ and quadratic mean $\mathcal{Q}(a, b)$ are, respectively, defined by

$$\mathcal{A}(a, b) = \frac{a+b}{2} \quad \text{and} \quad \mathcal{Q}(a, b) = \sqrt{\frac{a^2+b^2}{2}}.$$

It is well known that $\mathcal{A}(a, b) \leq \mathcal{Q}(a, b)$, and this inequality implies that

$$\frac{\|A\|_2 + |tr(A^2)|^{1/2}}{2} = \mathcal{A}(\|A\|_2, |tr(A^2)|^{1/2}) \leq \mathcal{Q}(\|A\|_2, |tr(A^2)|^{1/2}) = w_2(A). \quad \square \tag{5.2}$$

Observe that $\frac{1}{\sqrt{2}}\|A\|_2$ is not comparable to $\frac{\|A\|_2 + |tr(A^2)|^{1/2}}{2}$, as we see in the next examples:

1. If $A^2 = 0$ but $A \neq 0$, then $\frac{1}{\sqrt{2}}\|A\|_2 > \frac{\|A\|_2}{2}$.
2. Let $A = \begin{pmatrix} \alpha & 0 \\ 0 & i\beta \end{pmatrix}$, with $\alpha, \beta \in \mathbb{R}$. Then, $A^2 = \begin{pmatrix} \alpha^2 & 0 \\ 0 & -\beta^2 \end{pmatrix}$ and

$$\|A\|_2 + |tr(A^2)|^{1/2} = \sqrt{\alpha^2 + \beta^2} + \sqrt{|\alpha^2 - \beta^2|}.$$

In particular, if $\alpha = 1, \beta = 1 + \frac{1}{n}$ and $n = \frac{1}{10}$,

$$\left. \begin{aligned} \|A\|_2 + |tr(A^2)|^{1/2} &= \frac{\sqrt{122} + \sqrt{120}}{2} \approx 10.999 \\ \frac{1}{\sqrt{2}}\|A\|_2 &= \frac{\sqrt{122}}{\sqrt{2}} \approx 7.81. \end{aligned} \right\} \Rightarrow \|A\|_2 + |tr(A^2)|^{1/2} > \frac{1}{\sqrt{2}}\|A\|_2$$

On the other hand, if $n \in \mathbb{R}$ is chosen that $\left| \frac{2}{n} + \frac{1}{n^2} \right| < 0.001$, the reverse inequality holds.

3. Moreover, if any A fulfills $|tr(A^2)|^{1/2} > (\sqrt{2} - 1)\|A\|_2$, then

$$\sqrt{2}\|A\|_2 < |tr(A^2)|^{1/2} + \|A\|_2 \Rightarrow \frac{1}{\sqrt{2}}\|A\|_2 < \frac{|tr(A^2)|^{1/2} + \|A\|_2}{2}.$$

REMARK 2. Observe that if $A \neq 0$,

$$w_2(A) = \frac{|tr(A^2)|^{1/2} + \|A\|_2}{2} \Leftrightarrow |tr(A^2)|^{1/2} = \|A\|_2,$$

which means that

$$\frac{|tr(A^2)|^{1/2} + \|A\|_2}{2} = \|A\|_2 = w_2(A),$$

which occurs if and only if A is normal and the squares of its nonzero eigenvalues have the same argument (by Corollary 2 in [2]).

REMARK 3. Using (4.3) for a nonzero operator $A \in \mathcal{B}_2(H)$ we have

$$\frac{\|A\|_2^2 + |tr(A^2)|}{2} \leq \sup_{\theta \in \mathbb{R}} \int_0^1 \|((1-\lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*)\|_2^2 d\lambda \leq \|A\|_2^2. \tag{5.3}$$

but in this case the integral expression has an explicit formula, since

$$\begin{aligned} & \int_0^1 \|((1-\lambda)e^{i\theta}A + \lambda e^{-i\theta}A^*)\|_2^2 d\lambda \\ &= \int_0^1 \left[((1-\lambda)^2 + \lambda^2)\|A\|_2^2 + 2\lambda(1-\lambda)tr(Re(e^{2i\theta}A^2)) \right] d\lambda \\ &= \|A\|_2^2 \int_0^1 (2\lambda^2 - 2\lambda + 1) d\lambda + 2tr(Re(e^{2i\theta}A^2)) \int_0^1 \lambda - \lambda^2 d\lambda \\ &= \frac{2}{3}\|A\|_2^2 + \frac{1}{3}tr(Re(e^{2i\theta}A^2)) = \frac{2}{3}\|A\|_2^2 + \frac{1}{3} \left(2\|Re(e^{i\theta}A)\|_2^2 - \|A\|_2^2 \right) \\ &= \frac{1}{3}\|A\|_2^2 + \frac{2}{3}\|Re(e^{i\theta}A)\|_2^2 \end{aligned}$$

holds for any $\theta \in \mathbb{R}$. Then

$$\frac{\|A\|_2^2 + 3|tr(A^2)|}{4} \leq \|Re(e^{i\theta}A)\|_2^2 \leq \|A\|_2^2$$

and therefore

$$\frac{\sqrt{\|A\|_2^2 + 3|tr(A^2)|}}{2} \leq w_2(A) \leq \|A\|_2. \tag{5.4}$$

The lower bound in (5.4) is new but it is not better than $\frac{1}{\sqrt{2}}\|A\|_2$, since for $a, b \geq 0$ $\frac{1}{\sqrt{2}}a \leq \frac{\sqrt{a^2+3b^2}}{2}$ implies $a \leq b$.

In finite dimension, the attainment of the lower bound can be related with Birkhoff–James orthogonality, as it is shown in the following statement.

PROPOSITION 4. *Let $A \in \mathbb{M}_n$. Then the following conditions are equivalent*

1. $w_2(A) = \frac{1}{\sqrt{2}}\|A\|_2$.
2. $I \perp^p A^2$ for $1 \leq p < \infty$.

Moreover, if (1) and (2) are valid, then $I \perp \|\cdot\| \|A^2$ for any unitarily invariant norm $\|\cdot\|$.

Now, we focus in the general case when $p \in [1, \infty)$ and $p \neq 2$. From Theorem 2 in [2] it follows that

$$\frac{1}{2} \|A\|_p \leq w_p(A) \leq \|A\|_p \tag{5.5}$$

for any $1 \leq p < \infty$ and $A \in \mathcal{B}_p(H)$. Recall that if $1 \leq p_1 < p_2$ by (4) in Theorem 1, this implies that

$$w_{p_2}(T) \leq w_{p_1}(T). \tag{5.6}$$

We develop a number of inequalities to obtain new bounds for the generalized numerical radius w_p using the properties of the p -Schatten norms.

PROPOSITION 5. Let $A \in \mathcal{B}_p(H)$. Then

$$2^{-\frac{1}{p}} \|A\|_p \leq w_p(A) \leq \|A\|_p \tag{5.7}$$

for $1 \leq p \leq 2$, and

$$2^{\frac{1}{p}-1} \|A\|_p \leq w_p(A) \leq \|A\|_p \tag{5.8}$$

for $2 \leq p < \infty$.

Proof. We only prove the left hand side of (5.7). By Theorem 1 in [7] we have that for any $\theta \in \mathbb{R}$ hold

$$2^p \|A\|_p^p \leq 2^p (\|Re(e^{i\theta}A)\|_p^p + \|Im(e^{i\theta}A)\|_p^p) \leq 2^{p+1} w_p^p(A). \quad \square$$

Observe that in both cases, the lower bounds found improve $\frac{1}{2} \|A\|_p$. Also, note that if $p = 2$, we obtain Theorem 8 in [2].

PROPOSITION 6. Let $A \in \mathcal{B}_p(H)$, with $1 < p < \infty$, such that $w_p(A) = \|A\|_p$. Then $\|A^2\|_{p/2} = \|A\|_p^2$.

Proof. It is a direct consequence of Corollary 8 and Theorem 1. \square

Let $A \in \mathcal{B}_p(H)$ with $p \geq 1$, then for any $\theta \in \mathbb{R}$ we get $Im^2(e^{i\theta}A) = \frac{1}{2}(A^*A + AA^*) - Re^2(e^{i\theta}A) \geq 0$. It follows by Weyl’s monotonicity principle that if $S, T \in \mathcal{K}(H)$ are positive and $S \leq T$, then $s_j(S) \leq s_j(T)$ for any $j \in \mathbb{N}$, so $\|Re^2(e^{i\theta}A)\|_{p/2} \leq \frac{1}{2} \|A^*A + AA^*\|_{p/2}$. This shows that

$$w_p^2(A) \leq \frac{1}{2} \|A^*A + AA^*\|_{p/2} \tag{5.9}$$

To see that inequality (5.9) improves inequality (5.5) for $p \geq 2$, consider the chain of inequalities

$$w_p^2(A) \leq \frac{1}{2} \|A^*A + AA^*\|_{p/2} \leq \frac{1}{2} \|A^*A\|_{p/2} + \frac{1}{2} \|AA^*\|_{p/2} \leq \|A\|_p^2.$$

In the next theorem, we give a lower and upper bound for the generalized numerical radius w_p for $p \geq 2$.

THEOREM 7. *Let $A \in \mathcal{B}_p(H)$ with $p \geq 2$, then*

$$\begin{aligned} \sup_{\theta \in \mathbb{R}} \frac{\left| \|Re^2(e^{i\theta}A)\|_{p/2} - \|Im^2(e^{i\theta}A)\|_{p/2} \right|}{2} &\leq w_p^2(A) - \frac{1}{4} \|A^*A + AA^*\|_{p/2} \quad (5.10) \\ &\leq \frac{1}{2} w_{p/2}(A^2) \end{aligned}$$

Proof. First, observe that mimicking the idea in [36], by Theorem 2.3 we obtain for any $\theta \in \mathbb{R}$

$$\begin{aligned} w_p^2(A) &\geq \max\{\|Re(e^{i\theta}A)\|_p^2, \|Im^2(e^{i\theta}A)\|_p^2\} = \max\{\|Re^2(e^{i\theta}A)\|_{p/2}, \|Im^2(e^{i\theta}A)\|_{p/2}\} \\ &= \frac{\|Re^2(e^{i\theta}A)\|_{p/2} + \|Im^2(e^{i\theta}A)\|_{p/2}}{2} + \frac{\left| \|Re^2(e^{i\theta}A)\|_{p/2} - \|Im^2(e^{i\theta}A)\|_{p/2} \right|}{2} \\ &\geq \frac{1}{2} \|Re^2(e^{i\theta}A) + Im^2(e^{i\theta}A)\|_{p/2} + \frac{\left| \|Re^2(e^{i\theta}A)\|_{p/2} - \|Im^2(e^{i\theta}A)\|_{p/2} \right|}{2} \\ &= \frac{1}{4} \|A^*A + AA^*\|_{p/2} + \frac{\left| \|Re^2(e^{i\theta}A)\|_{p/2} - \|Im^2(e^{i\theta}A)\|_{p/2} \right|}{2}. \end{aligned}$$

Hence

$$\frac{1}{4} \|A^*A + AA^*\|_{p/2} + \sup_{\theta \in \mathbb{R}} \frac{\left| \|Re^2(e^{i\theta}A)\|_{p/2} - \|Im^2(e^{i\theta}A)\|_{p/2} \right|}{2} \leq w_p^2(A). \quad (5.11)$$

To prove the second inequality in (5.10), we have

$$\begin{aligned} w_p^2(A) &= \sup_{\theta \in \mathbb{R}} \|Re(e^{i\theta}A)\|_p^2 \\ &= \sup_{\theta \in \mathbb{R}} \frac{1}{4} \|e^{i\theta}A + e^{-i\theta}A^*\|_p^2 \\ &= \sup_{\theta \in \mathbb{R}} \frac{1}{4} \|(e^{i\theta}A + e^{-i\theta}A^*)^2\|_{p/2} \\ &= \sup_{\theta \in \mathbb{R}} \frac{1}{4} \|A^*A + AA^* + 2Re(e^{2i\theta}A^2)\|_{p/2} \\ &\leq \frac{1}{4} \|A^*A + AA^*\|_{p/2} + \frac{1}{2} \sup_{\theta \in \mathbb{R}} \|Re(e^{2i\theta}A^2)\|_{p/2} \\ &= \frac{1}{4} \|A^*A + AA^*\|_{p/2} + \frac{1}{2} w_{p/2}(A^2). \quad \square \end{aligned} \quad (5.12)$$

REMARK 4.

1. The inequalities in Theorem 7 can be extended to Q -norms. For more examples of these norms, the reader is referred to [5].
2. To see that left inequality in (5.10) improves Theorem 2.3 in [36], for p -Schatten norms with $p \geq 2$ we consider property (5.6).
3. Let $A \in \mathcal{B}_p(H)$ with $p \geq 2$. Considering the polar decomposition of A and Theorem 1 in [6] we have

$$s_j(A^2) = s_j(U|A|U|A|) \leq \frac{1}{2}s_j(A^*A + AA^*). \tag{5.13}$$

Thus

$$\begin{aligned} w_{p/2}(A^2) &\leq \|A^2\|_{p/2} = \left(\sum_{j=1}^{\infty} s_j(A^2)^{p/2} \right)^{2/p} \leq \frac{1}{2} \left(\sum_{j=1}^{\infty} s_j(A^*A + AA^*)^{p/2} \right)^{2/p} \\ &\leq \frac{1}{2} \|A^*A + AA^*\|_{p/2}, \end{aligned} \tag{5.14}$$

and hence Theorem 7 refines inequality (5.9).

For $0 < p < 1$, instead of the triangle inequality, which does not hold for the two-sided ideal $\mathcal{B}_p(H)$, we have $\|A+B\|_p^p \leq \|A\|_p^p + \|B\|_p^p$ for $A, B \in \mathcal{B}_p(H)$. Utilizing a similar argument as in Theorem 7 we get the following statement.

PROPOSITION 7. *Let $A \in \mathcal{B}_p(H)$ with $1 \leq p < 2$. Then*

$$\begin{aligned} \frac{1}{2^{p/2+1}} \|A^*A + AA^*\|_{p/2}^{p/2} + \sup_{\theta \in \mathbb{R}} \frac{\left| \|Re^2(e^{i\theta}A)\|_{p/2}^{p/2} - \|Im^2(e^{i\theta}A)\|_{p/2}^{p/2} \right|}{2} &\leq w_p^p(A) \\ &\leq \frac{1}{2^p} \|A^*A + AA^*\|_{p/2}^{p/2} + \frac{1}{2^{p/2}} w_{p/2}^{p/2}(A^2). \end{aligned} \tag{5.15}$$

Proof. First, observe that mimicking the idea in [36], by Theorem 2.3 we obtain for any $\theta \in \mathbb{R}$

$$\begin{aligned} w_p^p(A) &\geq \max\{\|Re(e^{i\theta}A)\|_p^p, \|Im(e^{i\theta}A)\|_p^p\} \\ &= \max\{\|Re^2(e^{i\theta}A)\|_{p/2}^{p/2}, \|Im^2(e^{i\theta}A)\|_{p/2}^{p/2}\} \\ &= \frac{\|Re^2(e^{i\theta}A)\|_{p/2}^{p/2} + \|Im^2(e^{i\theta}A)\|_{p/2}^{p/2}}{2} + \frac{\left| \|Re^2(e^{i\theta}A)\|_{p/2}^{p/2} - \|Im^2(e^{i\theta}A)\|_{p/2}^{p/2} \right|}{2} \\ &\geq \frac{1}{2} \|Re^2(e^{i\theta}A) + Im^2(e^{i\theta}A)\|_{p/2}^{p/2} + \frac{\left| \|Re^2(e^{i\theta}A)\|_{p/2}^{p/2} - \|Im^2(e^{i\theta}A)\|_{p/2}^{p/2} \right|}{2} \\ &= \frac{1}{2^{p/2+1}} \|A^*A + AA^*\|_{p/2}^{p/2} + \frac{\left| \|Re^2(e^{i\theta}A)\|_{p/2}^{p/2} - \|Im^2(e^{i\theta}A)\|_{p/2}^{p/2} \right|}{2}. \end{aligned}$$

Hence

$$\frac{1}{2^{p/2+1}} \|A^*A + AA^*\|_{p/2}^{p/2} + \sup_{\theta \in \mathbb{R}} \frac{\left| \|Re^2(e^{i\theta}A)\|_{p/2}^{p/2} - \|Im^2(e^{i\theta}A)\|_{p/2}^{p/2} \right|}{2} \leq w_p^p(A). \quad (5.16)$$

To prove the second inequality in (5.15), we have

$$\begin{aligned} w_p^p(A) &= \sup_{\theta \in \mathbb{R}} \|Re(e^{i\theta}A)\|_p^p \\ &= \sup_{\theta \in \mathbb{R}} \frac{1}{2^p} \|e^{i\theta}A + e^{-i\theta}A^*\|_p^p \\ &= \sup_{\theta \in \mathbb{R}} \frac{1}{2^p} \|(e^{i\theta}A + e^{-i\theta}A^*)^2\|_{p/2}^{p/2} \\ &= \sup_{\theta \in \mathbb{R}} \frac{1}{2^p} \|A^*A + AA^* + 2Re(e^{2i\theta}A^2)\|_{p/2}^{p/2} \\ &\leq \frac{1}{2^p} \|A^*A + AA^*\|_{p/2}^{p/2} + \frac{1}{2^{p/2}} \sup_{\theta \in \mathbb{R}} \|Re(e^{2i\theta}A^2)\|_{p/2}^{p/2} \\ &= \frac{1}{2^p} \|A^*A + AA^*\|_{p/2}^{p/2} + \frac{1}{2^{p/2}} w_{p/2}^{p/2}(A^2). \quad \square \end{aligned}$$

Since $(\mathcal{B}_p(H), \|\cdot\|_p)$ is a uniformly convex space for $1 < p < \infty$, we used the characterization of the norm-parallelism in p -Schatten ideals obtained in [13] (using the notion of semi-inner product in the sense of Lumer) and Theorem 5 to obtain the following statement for w_p .

PROPOSITION 8. *Let $A \in \mathcal{B}_p(H)$ with polar decompositions $A = U|A|$ and $A^* = V|A^*|$, respectively. If $1 < p < \infty$, then the following conditions are equivalent:*

1. $w_p(A) = \|A\|_p$.
2. $A\|A^*$.
3. $|\operatorname{tr}(|A|^{p-1}U^*A^*)| = \|A\|_p^p$.
4. $|\operatorname{tr}(|A^*|^{p-1}V^*A)| = \|A^*\|_p^p$.
5. $A = \alpha A^*$ with $\alpha \in \mathbb{T}$.

As a consequence of the preceding results and a combination of the different characterizations of norm parallelism for trace-class operators (in a context of a finite or infinite dimensional space) we obtain the following statement.

COROLLARY 3. *Let $A \in \mathcal{B}_1(H)$. Then the following conditions are equivalent:*

1. $w_1(A) = \|A\|_1$.

2. There exists $\theta \in \mathbb{R}$ such that $\|A + e^{i\theta}A^*\|_1 = \|A\|_1 + \|A^*\|_1$ (i.e. $A \parallel A^*$).
3. There exist a partial isometry V and $\lambda \in \mathbb{T}$ such that $A = V|A|$ and $A^* = \lambda V|A^*|$.
4. There exists $\lambda \in \mathbb{T}$ such that

$$\left| \text{tr}(|A|) + \lambda \text{tr}(U^*A^*) \right| \leq \left\| P_{\ker A^*}(A + \lambda A^*)P_{\ker A} \right\|_1,$$

where $A = U|A|$ is the polar decomposition of A .

5. $(A^*)^2 = \lambda|A||A^*|$
6. There exist isometries V, W and $\lambda \in \mathbb{T}$ such that

$$|A + \lambda A^*| = V|A|V^* + W|A^*|W^*.$$

7. There exists $\lambda \in \mathbb{T}$ such that $|A + \lambda A^*| = |A| + |A^*|$. (i.e. absolute value parallel definition in [34]).
8. There exist a closed subspace \mathcal{M} of H such that

$$\|A\|_1 = \text{tr}(P_{\mathcal{M}}e^{-i\frac{\theta}{2}}AP_{\mathcal{M}}) - \text{tr}(P_{\mathcal{M}^\perp}e^{-i\frac{\theta}{2}}AP_{\mathcal{M}^\perp}),$$

with $P_{\mathcal{M}}e^{-i\frac{\theta}{2}}AP_{\mathcal{M}} \geq 0$ and $P_{\mathcal{M}^\perp}e^{-i\frac{\theta}{2}}AP_{\mathcal{M}^\perp} \leq 0$.

If H is finite dimensional then (1) to (8) are also equivalent to

9. There exists $F \in \mathbb{M}_n$ such that $\|F\| \leq 1$, $\text{tr}(AF^*) = \|A\|_1 = |\text{tr}(A^*F^*)|$.

Furthermore if A is invertible, all the previous conditions are equivalent to

10. $\left| \text{tr}(|A|A^{-1}A^*) \right| = \|A^*\|_1$.

Proof. It is a direct consequence of Theorem 5, Theorem 2.15 and Corollary 2.16 in [34], Theorem 2.3 and 2.6 in [24] and Theorem 3.3 in [25]. \square

6. Inequalities involving w_N for products of operators

The following general result for the product of two operators holds, as a simple consequence of the upper bound of w_N with $N(\cdot)$ a selfadjoint norm on $\mathcal{B}(H)$: if $A, X \in \mathcal{B}(H)$ then

$$w_N(AX) \leq N(AX) \leq N(A)N(X) \leq 4w_N(A)w_N(X). \tag{6.1}$$

New estimates for the generalized numerical radius, w_N with $N(\cdot)$ a selfadjoint norm, of the product AX are given in this section. We begin with the following definitions which are necessities. Let $A, T \in \mathcal{B}(H)$, the vector-function $A - \lambda T$ is known as the

pencil generated by A and T . Evidently there is at least one complex number λ_0 such that

$$\|A - \lambda_0 T\| = \inf_{\lambda \in \mathbb{C}} \|A - \lambda T\|.$$

The number λ_0 is unique if $0 \notin \sigma_{ap}(T)$ (or equivalently if $\inf\{\|Tx\| : \|x\| = 1\} > 0$). The proofs of existence and unicity of such λ_0 are detailed in [28]. Different authors, following [32], called to this unique number as center of mass of A respect to T and we denote by $c(A, T)$ and when $T = I$ we write $c(A)$ and $D_A = \|A - c(A)I\| = \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|$.

Similarly for a norm $N(\cdot)$ we define

$$D_{N,A} = \inf_{\lambda \in \mathbb{C}} N(A - \lambda I).$$

With a similar proof as Theorem 1 in [28] we can state that $D_{N,A} = N(A - \lambda_0 I)$ for some $\lambda_0 \in \mathbb{C}$. If A, X are two bounded linear operators on H , then in [1] the authors proved

$$w(AX) \leq (\|A\| + D_A)w(X). \tag{6.2}$$

Since $D_A \leq \|A\|$, (6.2) is a refinement of the classical inequality $w(AX) \leq 2\|A\|w(X)$.

The following result is similar to (6.2) for the generalized numerical radius $w_N(\cdot)$. Here we give a proof for the convenience of the reader.

THEOREM 8. *For any $A, X \in \mathcal{B}(H)$ and $N(\cdot)$ a selfadjoint and submultiplicative norm on $\mathcal{B}(H)$ hold*

$$\begin{aligned} w_N(AX) &\leq \min \left\{ \frac{1}{2}w_N(AX \mp XA^*) + \frac{1}{2}w_N(AX \pm XA^*) \right\} \\ &\leq \min \left\{ N(A)w_N(X) + \frac{1}{2}w_N(AX \pm XA^*) \right\} \\ &\leq (N(A) + D_{N,A})w_N(X) \leq 2N(A)w_N(X) \leq 4w_N(A)w_N(X). \end{aligned} \tag{6.3}$$

Proof. To prove the first and second inequality we observe that $AX = \frac{1}{2}(AX \pm XA^*) + \frac{1}{2}(AX \mp XA^*)$ and we use the inequality (4) in Lemma 1. Now, we consider λ_0 such that $N(A - \lambda_0 I) = D_{N,A}$ and we write $\lambda_0 = e^{i\theta_0}|\lambda_0|$, then by the previous inequality we have

$$\begin{aligned} w_N(AX) &= w_N(e^{-i\theta_0}AX) \leq N(e^{-i\theta_0}A)w_N(X) + \frac{1}{2}w_N(e^{-i\theta_0}AX - e^{i\theta_0}XA^*) \\ &= N(e^{-i\theta_0}A)w_N(X) + \frac{1}{2}w_N(e^{-i\theta_0}(A - \lambda_0 I)X - e^{i\theta_0}X(A - \lambda_0 I)^*) \\ &\leq N(A)w_N(X) + N(e^{-i\theta_0}(A - \lambda_0 I))w_N(X) = (N(A) + D_{N,A})w_N(X). \end{aligned}$$

The fourth and fifth inequalities follows from the well-known facts: $D_{N,A} \leq N(A)$ and $N(A) \leq 2w_N(A)$. \square

COROLLARY 4. Let $A, X \in \mathcal{B}(H)$, $X \neq 0$ and $N(\cdot)$ a selfadjoint and submultiplicative norm on $\mathcal{B}(H)$. If $w_N(AX) = 2N(A)w_N(X)$ or $w_N(AX) = 4w_N(A)w_N(X)$ then $A \perp_N I$.

In order to obtain more refinements for the case $w_N = w_p$, we need the following results obtained by Bhatia and Zhan in [9, 10].

LEMMA 2. Let $T = A + iB \in \mathcal{B}_p(H)$ with $2 \leq p < \infty$ and suppose that A is positive and B is selfadjoint. Then

$$\|T\|_p^2 \leq \|A\|_p^2 + 2^{1-2/p}\|B\|_p^2. \tag{6.4}$$

Also when both A and B are positive then

$$\|T\|_p^2 \leq \|A\|_p^2 + \|B\|_p^2. \tag{6.5}$$

Recall that an operator $A \in \mathcal{B}(H)$ is called accretive if $Re(A) \geq 0$ and analogously A is dissipative if $Im(A) \geq 0$. Using these definitions, we have the following statement.

THEOREM 9. Let $A \in \mathcal{B}(H)$, $X \in \mathcal{B}_p(H)$ with $2 \leq p < \infty$ and X accretive. Then

$$w_p(AX) \leq \sqrt{1 + 2^{1-2/p}}\|A\|w_p(X). \tag{6.6}$$

Moreover, if X accretive and dissipative then

$$w_p(AX) \leq \sqrt{2}\|A\|w_p(X). \tag{6.7}$$

Proof. Since $\|\cdot\|_p$ is a selfadjoint norm, we recall that $w_p(AX) \leq \|AX\|_p \leq \|A\|\|X\|_p$. Now, if X is accretive then by Lemma 2, we have

$$\begin{aligned} w_p(AX) &\leq \|A\|\sqrt{\|Re(X)\|_p^2 + 2^{1-2/p}\|Im(X)\|_p^2} \\ &\leq \sqrt{1 + 2^{1-2/p}}\|A\|w_p(X). \end{aligned} \tag{6.8}$$

Furthermore, if X is also dissipative then

$$w_p(AX) \leq \|A\|\sqrt{\|Re(X)\|_p^2 + \|Im(X)\|_p^2} \leq \sqrt{2}\|A\|w_p(X). \quad \square \tag{6.9}$$

The following lemma plays a central role in our following statements.

LEMMA 3. ([23]) Let $A, X \in \mathcal{B}(H)$ such that AX is selfadjoint. If XA belongs to the norm ideal associated with a unitarily invariant norm $|||\cdot|||$, then AX belongs to this ideal and

$$|||AX||| \leq |||Re(XA)|||.$$

An important special case of the previous result asserts that if AX is selfadjoint and $AX \in \mathcal{B}_p(H)$ then

$$w_p(AX) = \|AX\|_p \leq \|Re(XA)\|_p \leq w_p(XA), \quad (6.10)$$

for $1 \leq p < \infty$.

THEOREM 10. *Let $A, X \in \mathcal{B}(H)$ such that AX is selfadjoint and $AX \in \mathcal{B}_p(H)$. If $XA \in \mathcal{B}_p(H)$ with $1 < p < \infty$. Then the following conditions are equivalent*

1. $w_p(AX) = w_p(XA)$.
2. $\|Re(XA)\|_p = \|AX\|_p$.
3. XA is selfadjoint.

Proof. (1) \Rightarrow (2) The equality follows from (6.10) and the hypothesis.

(2) \Leftrightarrow (3) The proof follows a similar idea to that of Kittaneh in [22] Lemma 2.

(3) \Rightarrow (1) We have the following inequality

$$w_p(AX) = \|AX\|_p = \|Re(AX)\|_p \geq \|XA\|_p = w_p(XA).$$

Combining with inequality (6.10), we obtain the equality desired. \square

Using Proposition 1 in [23] we obtain another lower bound of the numerical radius for the product of two positive operators. Then if $A, X \in \mathcal{B}(H)$

$$\|X^{1/2}AX^{1/2}\| = \|A^{1/2}X^{1/2}\|^2 \leq \|Re(AX)\| \leq w(AX)$$

and

$$\|X^{1/2}AX^{1/2}\|_{p/2} = \|A^{1/2}X^{1/2}\|_p^2 \leq \|Re(AX)\|_{p/2} \leq w_{p/2}(AX)$$

if $A, X \in \mathcal{B}_p(H)$ for $p \geq 2$.

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T. Bottazzi
Universidad Nacional de Río Negro, CITECCA
Sede Andina (8400) S.C. de Bariloche, Argentina
and
Consejo Nacional de Investigaciones Científicas y Técnicas
(1425) Buenos Aires, Argentina
e-mail: tbottazzi@unrn.edu.ar

C. Conde
Instituto Argentino de Matemática “Alberto P. Calderón”
Saavedra 15 3º piso, (C1083ACA) Buenos Aires, Argentina
and
Instituto de Ciencias
Universidad Nacional de Gral. Sarmiento
J. M. Gutierrez 1150, (B1613GSX) Los Polvorines, Argentina
e-mail: cconde@ungs.edu.ar