

THE A-MODEL WITH MUTUALLY EQUAL MODEL PARAMETERS CAN LEAD TO A HILBERT SPACE MODEL

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Abstract. It is known that the A-model for higher order singular perturbations can be considered as a Hilbert space model if the model parameters are mutually distinct, and that it is necessarily a Pontryagin space model if otherwise. In this note we demonstrate that the A-model with mutually equal model parameters can nonetheless lead to a Hilbert space model if the extensions in the model space are instead described by suitable linear relations.

1. Introduction

As it is known from [17], the A-model for rank one perturbations of class $\mathfrak{H}_{-m-2} \setminus \mathfrak{H}_{-m-1}$, $m \in \mathbb{N}$, of a lower semibounded self-adjoint operator L in \mathfrak{H}_0 is considered in general from the perspective of an indefinite inner product space (Pontryagin space), which we denote by \mathcal{H}_A . Here $(\mathfrak{H}_n, \langle \cdot, \cdot \rangle_n)_{n \in \mathbb{Z}}$ is the scale of Hilbert spaces associated with L , and the \mathfrak{H}_n -scalar product is defined via an operator $b_n(L) := \prod_{j=1}^n (L - z_j)$ with some fixed model parameters $z_j \in \text{res } L \cap \mathbb{R}$: $\langle \cdot, \cdot \rangle_n := \langle \cdot, b_n(L) \cdot \rangle_0$. The rank of indefiniteness of \mathcal{H}_A depends on the Gram matrix \mathcal{G}_A that determines an indefinite inner product $[\cdot, \cdot]_A$ in \mathcal{H}_A . By definition it is assumed that \mathcal{G}_A is invertible and Hermitian, but for perturbations of class \mathfrak{H}_{-4} or higher (i.e. $m \geq 2$), this is not sufficient in order to apply the extension theory of operators in \mathcal{H}_A . It appears that for such perturbations additional restrictions imposed on \mathcal{G}_A are needed; for example, for mutually equal model parameters z_j , the Gram matrix $\mathcal{G}_A = ([\mathcal{G}_A]_{jj'})$ must be of an anti-triangular form:

$$\begin{aligned} [\mathcal{G}_A]_{jj'} &= [\mathcal{G}_A]_{j'j} \in \mathbb{R}, & j, j' &\in \{1, \dots, m\}, \\ [\mathcal{G}_A]_{jj'} &= 0, & j &\in \{1, \dots, m-1\}, \quad j' \in \{1, \dots, m-j\}, \\ [\mathcal{G}_A]_{jm} &= [\mathcal{G}_A]_{j+1, m-1}, & j &\in \{1, \dots, m-1\}. \end{aligned} \tag{1.1}$$

More generally ([17, Theorem 3.2]), if at least two of the z_j 's are equal, then \mathcal{H}_A must have a nontrivial rank of indefiniteness; see also [11, Remark 4.10] with $z_j = 0$. In contrast, if the points z_j are all mutually distinct, then \mathcal{H}_A can be considered as a

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Hilbert space, *i.e.* there exists a positive matrix \mathcal{G}_A satisfying all necessary conditions required for the application of the theory of extensions to \mathcal{H}_A of L .

The main goal of this note is to demonstrate that, for equal z_j 's, we still can extract a Hilbert space model from the A-model provided that

$$[\mathcal{G}_A]_{mm} > 0, \quad [\mathcal{G}_A]_{m-1,m} = [\mathcal{G}_A]_{m,m-1} \in \mathbb{R} \tag{1.2}$$

for $m \geq 2$. In fact, we consider rank- d perturbations, with an arbitrary $d \in \mathbb{N}$, so that actually we have that $\mathcal{G}_A = ([\mathcal{G}_A]_{\sigma j, \sigma' j'})$ is a $dm \times dm$ Gram matrix; the indices σ, σ' range over an index set \mathcal{S} of cardinality $d \in \mathbb{N}$. The conditions in (1.1), (1.2) are then modified appropriately (see (2.5) and (3.1)).

In the A-model, singular perturbations of L in \mathcal{H}_A are specified by the extensions of a densely defined, closed, symmetric operator A_{\min} in \mathcal{H}_A , provided an invertible Hermitian \mathcal{G}_A satisfies appropriate conditions (for equal z_j 's these are as in (1.1)). We recall that A_{\min} is the adjoint in \mathcal{H}_A of the restriction $A_{\max} \supseteq A_{\min}$ to \mathcal{H}_A of the triplet adjoint L_{\max} of L_{\min} . The triplet adjoint is taken with respect to the Hilbert triple $\mathfrak{H}_m \subseteq \mathfrak{H}_0 \subseteq \mathfrak{H}_{-m}$. The operator L_{\min} is densely defined, closed, symmetric in \mathfrak{H}_m , has defect numbers (d, d) , and is essentially self-adjoint in \mathfrak{H}_0 , whose closure is L . As is usual in extension theory, an extension $A_\Theta \in \text{Ext}(A_{\min})$ is parametrized by a linear relation Θ in \mathbb{C}^d according to $\text{dom} A_\Theta = \{f \in \text{dom} A_{\max} \mid \Gamma f \in \Theta\}$, where $\Gamma := (\Gamma_0, \Gamma_1) : \text{dom} A_{\max} \rightarrow \mathbb{C}^d \times \mathbb{C}^d$ defines the boundary triple $(\mathbb{C}^d, \Gamma_0, \Gamma_1)$ for $A_{\max} = A_{\min}^*$.

To explain our main idea, let us now consider the A-model with equal model parameters, $z_j = z_1$. For simplicity we let $d = 1$. Let $\mathcal{H}_A^{\min} := \mathcal{H}_A \cap \mathfrak{H}_{m-2}$. The subscript “min”, indicating the minimality of the space, is due to the following fact. Because \mathcal{H}_A is the direct sum of \mathfrak{H}_m and an m -dimensional space \mathfrak{K}_A spanned by the singular elements $h_j \in \mathfrak{H}_{-m-2+2j} \setminus \mathfrak{H}_{-m-1+2j}$, we have that $\mathfrak{K}_A^{\min} \subseteq \mathfrak{K}_A \subseteq \mathfrak{H}_{-m}$, where $\mathfrak{K}_A^{\min} := \mathfrak{K}_A \cap \mathfrak{H}_{m-2}$ is a minimal subset contained in \mathfrak{K}_A in the sense that $\mathfrak{K}_A \cap \mathfrak{K}_{m-1} = \{0\}$.

Consider the domain restriction $A_{\max} \upharpoonright_{\mathcal{H}_A^{\min}}$ to $\mathcal{H}_A^{\min} = \mathfrak{H}_m \dot{+} \mathfrak{K}_A^{\min}$ of A_{\max} . Let B_{\max} denote a linear relation in \mathcal{H}_A defined by the componentwise sum of (the graph of) $A_{\max} \upharpoonright_{\mathcal{H}_A^{\min}}$ and $\{0\} \times \mathcal{H}_A^\perp$. Here \mathcal{H}_A^\perp denotes the orthogonal complement in \mathcal{H}_A of \mathcal{H}_A^{\min} , which is a subset of \mathfrak{K}_A . By the construction, the adjoint $B_{\min} := B_{\max}^*$ in \mathcal{H}_A is a linear relation given by the componentwise sum of (the graph of) $A_{\min} \upharpoonright_{\mathcal{H}_A^{\min}}$ and $\{0\} \times \mathcal{H}_A^\perp$. Assuming only the invertibility and the Hermiticity of \mathcal{G}_A , the operator A_{\min} differs from $A'_{\min} := A_{\max} \upharpoonright_{\ker \Gamma}$ (although $\text{dom} A_{\min} = \text{dom} A'_{\min}$), *i.e.* A_{\min} is not symmetric; the symmetry of $A_{\min} = A'_{\min}$ is ensured by (1.1). Now the key point is that, without assumption (1.1), but instead assuming $[\mathcal{G}_A]_{m-1,m} = [\mathcal{G}_A]_{m,m-1}$ (the second condition in (1.2)), it holds

$$(A_{\min} - A'_{\min})(\text{dom} A_{\min} \cap \mathcal{H}_A^{\min}) \subseteq \mathcal{H}_A^\perp$$

i.e. B_{\min} is a symmetric linear relation in \mathcal{H}_A . By the same reasoning one shows that B_{\min} is also closed. Sequentially, one can apply the extension theory for B_{\min} , as is done for A_{\min} .

For \mathcal{G}_A as in (1.1), the Weyl function corresponding to a boundary triple for A_{\max} determined by Γ is the sum of the Krein Q -function q of L_{\min} and a generalized Nevanlinna function r (see e.g. [2, Section 4] for the terminology) defined by

$$r(z) := - \sum_{j=1}^m \frac{[\mathcal{G}_A]_{mj}}{(z - z_1)^{m-j+1}}, \quad z \in \mathbb{C} \setminus \{z_1\}.$$

Likewise, for \mathcal{G}_A as in (1.2), the Weyl function corresponding to the boundary triple for B_{\max} , which is determined by restriction to $\text{dom } B_{\max}$ of Γ , is the sum of the same Krein Q -function q and now a Nevanlinna function \hat{r} defined by

$$\hat{r}(z) := \frac{[\mathcal{G}_A]_{mm}}{\hat{\Delta} - z}, \quad z \in \mathbb{C} \setminus \{\hat{\Delta}\}$$

with some real number $\hat{\Delta}$. The strict inequality $[\mathcal{G}_A]_{mm} > 0$ in (1.2) is closely related to the fact that the subspace $\mathcal{H}_A^{\min} = (\mathfrak{H}_m \dot{+} \mathfrak{K}_A^{\min}, [\cdot, \cdot]_A)$ of \mathcal{H}_A is a Hilbert space iff $[\mathcal{G}_A]_{mm} > 0$. Thus, for example, one may take \mathcal{G}_A as the Gram matrix of vectors h_j generating \mathfrak{K}_A , in which case $[\mathcal{G}_A]_{jj'} = \langle h_j, h_{j'} \rangle_{-m}$, and the conditions in (1.2) are all satisfied. In contrast, the so defined \mathcal{G}_A does not satisfy (1.1). We remark that, for $m = 1$, we have $\hat{\Delta} = z_1$, and hence $\hat{r} = r$, as it should follow from $\mathcal{H}_A^{\min} = \mathcal{H}_A$. We also remark that an analogous development of extension theory for B_{\min} takes place in the peak model for singular perturbations, cf. [22].

Because the Weyl function $q + \hat{r}$ of B_{\min} is a (uniformly strict) Nevanlinna function, it follows from [24, Theorem 2.2] that $q + \hat{r}$ is the Weyl function of some closed simple symmetric operator, corresponding to a certain boundary triple. Following the terminology in [18], it is precisely in this sense what we mean by saying that the A-model with mutually equal model parameters leads to a Hilbert space model (of the function $q + \hat{r}$). For example, a simple symmetric operator may be considered as the operator of multiplication by an independent variable in a reproducing kernel Hilbert space induced by the Nevanlinna pair $(1, q + \hat{r})$; see e.g. [3, Theorem 6.1], [2, Theorem 4.10], [9, Remark 2.6].

Having determined the extensions to \mathcal{H}_A of L_{\min} one then interprets singular perturbations of L by means of the compressions to \mathfrak{H}_m of their resolvents. Thus, for $d = 1$, $B_{\Theta} \in \text{Ext}(B_{\min})$, $\Theta \in \mathbb{C} \cup \{\infty\}$, the compressed resolvent of B_{Θ} is represented in the generalized sense according to

$$P_{\mathfrak{H}_m}(B_{\Theta} - z)^{-1} |_{\mathfrak{H}_m} = (L - z)^{-1} + \frac{\langle g(\bar{z}), \cdot \rangle (L - z)^{-1} h_m}{\Theta - q(z) - \hat{r}(z)}$$

for a suitable $z \in \text{res } L$. Here $P_{\mathfrak{H}_m}$ is a projection in \mathcal{H}_A onto \mathfrak{H}_m , $g(\bar{z}) \in \mathfrak{H}_{-m} \setminus \mathfrak{H}_{-m+1}$ is the eigenvector of L_{\max} corresponding to the eigenvalue \bar{z} (in particular $h_1 = g(z_1)$), and $\langle \cdot, \cdot \rangle$ is the duality pairing between \mathfrak{H}_{-m} and \mathfrak{H}_m . By the above resolvent formula one concludes that the spectral properties of (super) singular perturbations in the A-model with equal model parameters can be described by Nevanlinna functions.

The reasoning behind the above mentioned interpretation of singular perturbations is that there exists a bijective correspondence between Nevanlinna families and generalized resolvents of L_{\min} , and the correspondence is established via a generalized

Krein–Naimark resolvent formula. Thus, to a rational Nevanlinna function $\hat{r} - \Theta$, with a real Θ , there corresponds a self-adjoint extension \tilde{B} of L_{\min} in some larger Hilbert space $\tilde{\mathfrak{H}} \supseteq \mathfrak{H}_m$, and such that $\tilde{B} \cap L = L_{\min}$. For more details the reader may refer to [4, 18, 13, 8, 7, 14].

2. A brief overview of the A-model with equal model parameters

Here we restate the main results from [23, 17]; see also [11]. The main tools and terminology used in the theory of boundary relations of symmetric operators (or linear relations) are as in [4, 15, 10, 20, 16, 2, 19, 21, 8] and in references therein.

We consider a lower semibounded self-adjoint operator L in a Hilbert space \mathfrak{H}_0 , and we let $(\mathfrak{H}_n)_{n \in \mathbb{Z}}$ be the scale of Hilbert spaces associated with L . The scalar product in \mathfrak{H}_n is conjugate linear in the first factor and is defined via the scalar product $\langle \cdot, \cdot \rangle_0$ in \mathfrak{H}_0 according to

$$\langle \cdot, \cdot \rangle_n := \langle b_n(L)^{1/2} \cdot, b_n(L)^{1/2} \cdot \rangle_0, \quad b_n(L) := (L - z_1)^n$$

for some fixed model parameter $z_1 \in \text{res } L \cap \mathbb{R}$ ($\text{res } L$ denotes the resolvent set of L , and similarly for other operators). To $L = L_0$ one associates a self-adjoint operator $L_n := L|_{\mathfrak{H}_{n+2}}$ in \mathfrak{H}_n , and satisfying $L_{n+1} \subset L_n$ and $\text{res } L_n = \text{res } L$. For the reasons just described we sometimes omit the subscript n in L_n .

Let us fix $m, d \in \mathbb{N}$. Let $\{\varphi_\sigma \in \mathfrak{H}_{-m-2} \setminus \mathfrak{H}_{-m-1}\}$ be the family of linearly independent functionals; σ ranges over an index set \mathcal{S} of cardinality d . The symmetric restriction L_{\min} of L to the domain of $f \in \mathfrak{H}_{m+2}$ such that $\langle \varphi_\sigma, f \rangle = 0$, for all σ , is a densely defined, closed, symmetric operator in \mathfrak{H}_m , and has defect numbers (d, d) . It is also essentially self-adjoint operator in \mathfrak{H}_0 . The duality pairing $\langle \cdot, \cdot \rangle$ is defined via the \mathfrak{H}_0 -scalar product in a usual way. We also define a vector valued functional φ via $\langle \varphi, \cdot \rangle = (\langle \varphi_\sigma, \cdot \rangle) : \mathfrak{H}_{m+2} \rightarrow \mathbb{C}^d$; hence $L_{\min} = L_m|_{\{f \in \mathfrak{H}_{m+2} \mid \langle \varphi, f \rangle = 0\}}$.

The triplet adjoint L_{\max} of L_{\min} corresponding to the Hilbert triple $\mathfrak{H}_m \subset \mathfrak{H}_0 \subset \mathfrak{H}_{-m}$ is the operator extending L_{-m} to the domain $\mathfrak{H}_{-m-2} \dot{+} \mathfrak{N}_z(L_{\max})$ (direct sum) for $z \in \text{res } L$. The eigenspace $\mathfrak{N}_z(L_{\max})$ ($:= \ker(L_{\max} - z)$) is the linear span of the elements $g_\sigma(z)$ defined in the generalized sense according to

$$g_\sigma(z) := (L - z)^{-1} \varphi_\sigma \in \mathfrak{H}_{-m} \setminus \mathfrak{H}_{-m+1}.$$

Define an md -dimensional linear space

$$\mathfrak{K}_A := \text{span}\{h_\alpha \mid \alpha = (\sigma, j) \in \mathcal{S} \times J\}, \quad J := \{1, 2, \dots, m\}$$

spanned by the elements

$$h_{\sigma j} := b_j(L)^{-1} \varphi_\sigma \in \mathfrak{H}_{-m-2+2j} \setminus \mathfrak{H}_{-m-1+2j}.$$

From here it follows that $\mathfrak{K}_A^{\min} \subseteq \mathfrak{K}_A \subseteq \mathfrak{H}_{-m}$ with

$$\mathfrak{K}_A^{\min} := \mathfrak{K}_A \cap \mathfrak{H}_{m-2} = h_m(\mathbb{C}^d), \quad h_m(c) := \sum_{\sigma} c_\sigma h_{\sigma m}, \quad c = (c_\sigma) \in \mathbb{C}^d$$

and that in particular $\mathfrak{K}_A^{\min} = \mathfrak{K}_A$ for $m = 1$. Note that $\mathfrak{K}_A \cap \mathfrak{H}_{m-1} = \{0\}$.

Because the system $\{h_\alpha\}$ is linearly independent, the matrix

$$\tilde{\mathcal{G}}_A = ([\tilde{\mathcal{G}}_A]_{\alpha\alpha'}) \in [\mathbb{C}^{md}], \quad [\tilde{\mathcal{G}}_A]_{\alpha\alpha'} := \langle h_\alpha, h_{\alpha'} \rangle_{-m}$$

is the Gram matrix of vectors generating \mathfrak{K}_A ; hence it is positive definite, Hermitian. One establishes a bijective correspondence

$$\mathfrak{K}_A \ni k \leftrightarrow d(k) = (d_\alpha(k)) \in \mathbb{C}^{md}$$

via

$$k = \sum_{\alpha} d_\alpha(k) h_\alpha, \quad d(k) = \tilde{\mathcal{G}}_A^{-1} \langle h, k \rangle_{-m}, \quad \langle h, \cdot \rangle_{-m} = (\langle h_\alpha, \cdot \rangle_{-m}).$$

Here and in what follows $d(\cdot)$ is interpreted as a (bounded) vector valued functional from \mathfrak{K}_A to \mathbb{C}^{md} .

Let us define the matrix

$$\tilde{\mathcal{G}}_A^{\min} = ([\tilde{\mathcal{G}}_A^{\min}]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [\tilde{\mathcal{G}}_A^{\min}]_{\sigma\sigma'} := \langle h_{\sigma m}, h_{\sigma' m} \rangle_{-m}$$

which is the Gram matrix of vectors generating \mathfrak{K}_A^{\min} . Thus $\tilde{\mathcal{G}}_A^{\min}$ is also positive definite, Hermitian, and one therefore establishes a bijective correspondence

$$\mathfrak{K}_A^{\min} \ni h_m(c) \leftrightarrow c \in \mathbb{C}^d$$

via

$$c = (\tilde{\mathcal{G}}_A^{\min})^{-1} \langle h_m, h_m(c) \rangle_{-m}, \quad \langle h_m, \cdot \rangle_{-m} = (\langle h_{\sigma m}, \cdot \rangle_{-m}).$$

On the other hand, because $\mathfrak{K}_A^{\min} \subseteq \mathfrak{K}$, to each $k = h_m(c) \in \mathfrak{K}_A^{\min}$ there corresponds $d(k) = \eta(c) \in \mathbb{C}^{md}$, where

$$\eta(c) := (\delta_{jm} c_\sigma).$$

Consider an indefinite inner product space

$$\mathcal{H}_A := (\mathfrak{H}_m \dot{+} \mathfrak{K}_A, [\cdot, \cdot]_A)$$

equipped with an indefinite metric

$$[f + k, f' + k']_A := \langle f, f' \rangle_m + \langle d(k), \mathcal{G}_A d(k') \rangle_{\mathbb{C}^{md}}$$

for $f, f' \in \mathfrak{H}_m$ and $k, k' \in \mathfrak{K}_A$. The matrix $\mathcal{G}_A = ([\mathcal{G}_A]_{\alpha\alpha'})$ is called the Gram matrix of the A-model; it is initially assumed to be invertible and Hermitian, but otherwise arbitrary. Thus in particular $\mathcal{G}_A \neq 0$. Clearly if \mathcal{G}_A is positive, then \mathcal{H}_A becomes a Hilbert space. Otherwise \mathcal{H}_A is a Pontryagin space.

For an appropriate \mathcal{G}_A , the extensions to \mathcal{H}_A of L_{\min} are the restrictions to \mathcal{H}_A of the triplet adjoint L_{\max} . Let

$$A_{\max} := L_{\max} \cap \mathcal{H}_A^2.$$

Here and in what follows operators are frequently identified with their graphs. The operator A_{\max} admits the following representation:

$$A_{\max} = \{ (f^\# + h_{m+1}(c) + k, L_m f^\# + z_1 h_{m+1}(c) + \tilde{k}) \mid f^\# \in \mathfrak{H}_{m+2}; \\ c \in \mathbb{C}^d; k, \tilde{k} \in \mathfrak{K}_A; d(\tilde{k}) = \mathfrak{M}_d d(k) + \eta(c) \}.$$

An element $h_{m+1}(c) \in \mathfrak{H}_m \setminus \mathfrak{H}_{m+1}$ is defined by

$$h_{m+1}(c) := \sum_{\sigma} c_{\sigma} h_{\sigma, m+1}, \quad h_{\sigma, m+1} := b_{m+1}(L)^{-1} \varphi_{\sigma}.$$

The matrix $\mathfrak{M}_d := \mathfrak{M} \oplus \dots \oplus \mathfrak{M}$ (d times) is the matrix direct sum of d matrices $\mathfrak{M} = (\mathfrak{M}_{jj'}) \in [\mathbb{C}^m]$ defined as follows: For $m \geq 2$

$$\mathfrak{M}_{jj'} := 1_{J \setminus \{m\}}(j) (\delta_{jj'} z_1 + 1_{J \setminus \{1\}}(j') \delta_{j+1, j'}) + \delta_{jm} \delta_{j'm} z_1$$

for $j, j' \in J$; here 1_X is the characteristic function of a set X . For $m = 1$, $\mathfrak{M} := z_1$.

By direct computation, the boundary form of A_{\max} is represented in the form

$$[f, A_{\max} g]_A - [A_{\max} f, g]_A = \langle d(k), (\mathcal{G}_{\mathfrak{M}} - \mathcal{G}_{\mathfrak{M}}^*) d(k') \rangle_{\mathbb{C}^{md}} \\ + \langle \Gamma_0 f, \Gamma_1 g \rangle_{\mathbb{C}^d} - \langle \Gamma_1 f, \Gamma_0 g \rangle_{\mathbb{C}^d},$$

$$\mathcal{G}_{\mathfrak{M}} := \mathcal{G}_A \mathfrak{M}_d$$

with $f = f^\# + h_{m+1}(c) + k \in \text{dom } A_{\max}$; $g = g^\# + h_{m+1}(c') + k' \in \text{dom } A_{\max}$; $f^\#, g^\# \in \mathfrak{H}_{m+2}$; $c, c' \in \mathbb{C}^d$; $k, k' \in \mathfrak{K}_A$. The operator $\Gamma := (\Gamma_0, \Gamma_1)$ from $\text{dom } A_{\max}$ to $\mathbb{C}^d \times \mathbb{C}^d$ is defined by

$$\Gamma_0(f^\# + h_{m+1}(c) + k) := c, \\ \Gamma_1(f^\# + h_{m+1}(c) + k) := \langle \varphi, f^\# \rangle - [\mathcal{G}_A d(k)]_m$$

with

$$[\mathcal{G}_A d(k)]_m := ([\mathcal{G}_A d(k)]_{\sigma m}) \in \mathbb{C}^d.$$

In the next lemma we give a description of the adjoint of A_{\max} and, moreover, we show that Γ is surjective. By considering Γ as a single-valued linear relation from \mathcal{H}_A^2 to \mathbb{C}^{2d} with $\text{dom } \Gamma = A_{\max}$, *i.e.*

$$\Gamma = \{ ((f, A_{\max} f), (\Gamma_0 f, \Gamma_1 f)) \mid f \in \text{dom } A_{\max} \}$$

we recall (*e.g.* [2, Section 3.1]) that its Krein space adjoint $\Gamma^{[*]}$ is a linear relation from \mathbb{C}^{2d} to \mathcal{H}_A^2 , and it consists of $((\mathcal{X}, \mathcal{X}'), (g, g'))$ such that $(\forall f \in \text{dom } A_{\max})$

$$[f, g']_A - [A_{\max} f, g]_A = \langle \Gamma_0 f, \mathcal{X}' \rangle_{\mathbb{C}^d} - \langle \Gamma_1 f, \mathcal{X} \rangle_{\mathbb{C}^d}. \tag{2.1}$$

LEMMA 2.1. *Similar to A_{\max} , define the operator A'_{\max} in \mathcal{H}_A by*

$$A'_{\max} := \{ (f^\# + h_{m+1}(c) + k, L_m f^\# + z_1 h_{m+1}(c) + \tilde{k}') \mid f^\# \in \mathfrak{H}_{m+2}; \\ c \in \mathbb{C}^d; k, \tilde{k}' \in \mathfrak{K}_A; d(\tilde{k}') = \mathcal{G}_A^{-1} \mathcal{G}_{\mathfrak{M}}^* d(k) + \eta(c) \}.$$

The following statements hold:

(i) Consider $\Gamma = (\Gamma_0, \Gamma_1)$ as a single-valued linear relation with $\text{dom}\Gamma = A_{\max}$. Let $\Gamma^{[*]}$ be its Krein space adjoint. Then the inverse $(\Gamma^{[*]})^{-1} = (\Gamma_0, \Gamma_1)$ is a single-valued linear relation with $\text{ran}\Gamma^{[*]} = A'_{\max}$. Moreover, Γ is closed and surjective.

(ii) The adjoint in \mathcal{H}_A of a closed operator A_{\max} is the operator

$$A_{\min} := A_{\max}^* = A'_{\max} \upharpoonright_{\ker \Gamma}.$$

(iii) Define the operator

$$A'_{\min} := A_{\max} \upharpoonright_{\ker \Gamma}$$

in \mathcal{H}_A . Then A'_{\min} is closed, and its adjoint in \mathcal{H}_A is $A'^{*}_{\min} = A'_{\max}$.

Proof. First we remark that

$$\text{ran } \mathcal{G}_{\mathfrak{M}}^* \subseteq \text{ran } \mathcal{G}_A \tag{2.2}$$

so that A'_{\max} is defined correctly. The inclusion in (2.2) is equivalent to the statement that

$$(\forall \xi \in \mathbb{C}^{md}) (\exists \xi' \in \mathbb{C}^{md}) \mathcal{G}_{\mathfrak{M}}^* \xi = \mathcal{G}_A \xi'. \tag{2.3}$$

For $m = 1$, $\mathcal{G}_{\mathfrak{M}} = z_1 \mathcal{G}_A$, so $\xi' = z_1 \xi$ solves (2.3) for an arbitrary Hermitian \mathcal{G}_A . For $m \geq 2$ we have

$$[\mathcal{G}_{\mathfrak{M}}]_{\sigma_j, \sigma'_{j'}} = z_1 [\mathcal{G}_A]_{\sigma_j, \sigma'_{j'}} + 1_{J \setminus \{1\}}(j') [\mathcal{G}_A]_{\sigma_j; \sigma', j'-1}$$

and hence

$$[\mathcal{G}_{\mathfrak{M}}^*]_{\sigma_j, \sigma'_{j'}} = z_1 [\mathcal{G}_A]_{\sigma_j, \sigma'_{j'}} + 1_{J \setminus \{1\}}(j) [\mathcal{G}_A]_{\sigma, j-1; \sigma'_{j'}}.$$

Then $\mathcal{G}_{\mathfrak{M}}^* \xi = \mathcal{G}_A \xi'$ reads

$$[\mathcal{G}_A(\xi' - z_1 \xi)]_{\sigma_j} = 1_{J \setminus \{1\}}(j) [\mathcal{G}_A \xi]_{\sigma, j-1}.$$

Put

$$\mathring{\mathcal{G}}_A = ((\mathring{\mathcal{G}}_A)_{\alpha\alpha'}) \in [\mathbb{C}^{md}], \quad [\mathring{\mathcal{G}}_A]_{\sigma_j, \sigma'_{j'}} := 1_{J \setminus \{1\}}(j) [\mathcal{G}_A]_{\sigma, j-1; \sigma'_{j'}}.$$

Then

$$\text{ran } \mathring{\mathcal{G}}_A \subseteq \text{ran } \mathcal{G}_A \quad \text{and} \quad \mathcal{G}_A(\xi' - z_1 \xi) = \mathring{\mathcal{G}}_A \xi$$

and therefore

$$\xi' = (z_1 + \mathcal{G}_A^{-1} \mathring{\mathcal{G}}_A) \xi$$

solves (2.3) for an arbitrary invertible Hermitian \mathcal{G}_A .

(i) Letting $g = g^{\natural} + k_g$ and $g' = g'^{\natural} + k_{g'}$ in (2.1) for some $g^{\natural}, g'^{\natural} \in \mathfrak{H}_m$ and $k_g, k_{g'} \in \mathfrak{K}_A$, and using that

$$\langle \langle \varphi, f^{\sharp} \rangle, \mathcal{X} \rangle_{\mathbb{C}^d} = \langle (L - z_1) f^{\sharp}, h_{m+1}(\mathcal{X}) \rangle_m$$

and

$$\langle [\mathcal{G}_A d(k)]_m, \chi \rangle_{\mathbb{C}^d} = \langle d(k), \mathcal{X}\chi \rangle_{\mathbb{C}^{md}}$$

with

$$\mathcal{X} = ([\mathcal{X}]_{\alpha\sigma}) \in [\mathbb{C}^d, \mathbb{C}^{md}], \quad [\mathcal{X}]_{\alpha\sigma} := [\mathcal{G}_A]_{\alpha, \sigma'_m}$$

we find that $(\forall f^\# \in \mathfrak{H}_{m+2}) (\forall c \in \mathbb{C}^d) (\forall k \in \mathfrak{K}_A)$

$$\begin{aligned} 0 = & \langle f^\#, g'^{\natural} - z_1 h_{m+1}(\chi) \rangle_m - \langle Lf^\#, g^{\natural} - h_{m+1}(\chi) \rangle_m \\ & + \langle c, \langle h_{m+1}, g'^{\natural} - z_1 g^{\natural} \rangle_m - [\mathcal{G}_A d(k_g)]_m - \mathcal{X}' \rangle_{\mathbb{C}^d} \\ & + \langle d(k), \mathcal{G}_A d(k_{g'}) - \mathcal{G}_{\mathfrak{M}}^* d(k_g) - \mathcal{X}\chi \rangle_{\mathbb{C}^{md}} \end{aligned} \tag{2.4}$$

with

$$\langle h_{m+1}, \cdot \rangle_m = (\langle h_{\sigma, m+1}, \cdot \rangle_m).$$

Thus it follows that

$$g^{\natural} = g^\# + h_{m+1}(\chi), \quad g^\# \in \mathfrak{H}_{m+2}, \quad g'^{\natural} = L_m g^\# + z_1 h_{m+1}(\chi)$$

and

$$\mathcal{X}' = \langle h_{m+1}, (L - z_1)g^\# \rangle_m - [\mathcal{G}_A d(k_g)]_m = \langle \varphi, g^\# \rangle - [\mathcal{G}_A d(k_g)]_m$$

and

$$d(k_{g'}) = \mathcal{G}_A^{-1} \mathcal{G}_{\mathfrak{M}}^* d(k_g) + \mathcal{G}_A^{-1} \mathcal{X}\chi, \quad \mathcal{G}_A^{-1} \mathcal{X}\chi = \eta(\chi).$$

This shows that

$$(\Gamma^{[*]})^{-1} = \{((f, A'_{\max} f), (\Gamma_0 f, \Gamma_1 f)) \mid f \in \text{dom } A_{\max}\}.$$

Because $\ker \Gamma^{[*]} = \text{mul}(\Gamma^{[*]})^{-1} = \{0\}$, it follows that $\overline{\text{ran}} \Gamma = \text{ran } \overline{\Gamma} = \mathbb{C}^{2d}$, and it therefore remains to verify that Γ is closed.

The closure $\overline{\Gamma}$ is the Krein space adjoint of $\Gamma^{[*]}$. Thus it consists $((g, g'), (\chi, \chi')) \in \mathcal{H}_A^2 \times \mathbb{C}^{2d}$ such that $(\forall f \in \text{dom } A_{\max})$ equation (2.4) holds, but with $\mathcal{G}_{\mathfrak{M}}^*$ replaced by $\mathcal{G}_{\mathfrak{M}}$. By repeating the subsequent steps as above, one finds that $\overline{\Gamma} = \Gamma$.

(ii) The adjoint linear relation A_{\min} consists of $(g, g') \in \mathcal{H}_A^2$ such that (2.4) holds, but with $\chi = 0 = \chi'$; therefore it is the operator as stated in the lemma.

(iii) By the arguments as in the proof of (i), $A_{\max}' = A_{\min}'$; thus A_{\min}' is a closed operator whose adjoint in \mathcal{H}_A is as stated in the lemma. \square

For $m = 1$, the matrix $\mathcal{G}_{\mathfrak{M}} = z_1 \mathcal{G}_A$ is automatically Hermitian, while for $m \geq 2$, we have $\mathcal{G}_{\mathfrak{M}}^* = \mathcal{G}_{\mathfrak{M}}$ iff

$$\begin{aligned} [\mathcal{G}_A]_{\sigma j, \sigma' j'} &= [\mathcal{G}_A]_{\sigma j', \sigma' j}, \quad j, j' \in J, \\ [\mathcal{G}_A]_{\sigma j, \sigma' j'} &= 0, \quad j \in J \setminus \{m\}, \quad j' \in \{1, \dots, m-j\}, \\ [\mathcal{G}_A]_{\sigma j, \sigma' m} &= [\mathcal{G}_A]_{\sigma, j+1; \sigma', m-1}, \quad j \in J \setminus \{m\}. \end{aligned} \tag{2.5}$$

Note that the entries of \mathcal{G}_A in (2.5), which are diagonal in $\sigma \in \mathcal{S}$, are real numbers. Note also that $\tilde{\mathcal{G}}_A$ does not satisfy (2.5), because $[\tilde{\mathcal{G}}_A]_{\sigma 1, \sigma 1} > 0$.

For an Hermitian $\mathcal{G}_{\mathfrak{M}}$ we have $A'_{\max} = A_{\max}$, $A'_{\min} = A_{\min}$, and Γ is a unitary operator, $\Gamma^{-1} = \Gamma^{[*]}$. Subsequently [12, Corollary 2.4], the triple $(\mathbb{C}^d, \Gamma_0, \Gamma_1)$ is a boundary triple for the adjoint $A_{\max} = A_{\min}^*$ of a densely defined, closed, and symmetric operator A_{\min} in a Pontryagin space \mathcal{H}_A ; the reader may refer to [8, Definition 2.1], [7, Definition 3] for the formal definition of a boundary triple. An extension $A_{\Theta} \in \text{Ext}(A_{\min})$ of A_{\min} , *i.e.* an operator satisfying $A_{\min} \subseteq A_{\Theta} \subseteq A_{\max}$, is parametrized by a linear relation Θ in \mathbb{C}^d according to

$$\text{dom}A_{\Theta} = \{f \in \text{dom}A_{\max} \mid \Gamma f \in \Theta\}.$$

In particular, A_{Θ} is self-adjoint in \mathcal{H}_A iff Θ is self-adjoint in \mathbb{C}^d , because the adjoint A_{Θ}^* in \mathcal{H}_A of A_{Θ} is given by A_{Θ^*} , where Θ^* is the adjoint in \mathbb{C}^d of Θ . The Krein–Naimark resolvent formula for A_{Θ} reads ([7, Theorem 2(iii)])

$$(A_{\Theta} - z)^{-1} = (A_0 - z)^{-1} + \gamma_{\Gamma}(z)(\Theta - M_{\Gamma}(z))^{-1}\gamma_{\Gamma}(z)^*$$

for $z \in \text{res}A_0 \cap \text{res}A_{\Theta}$. The self-adjoint operator A_0 corresponds to the self-adjoint linear relation $\{0\} \times \mathbb{C}^d$ in \mathbb{C}^d , and its resolvent is given by

$$(A_0 - z)^{-1}(f + k) = (L_m - z)^{-1}f + \sum_{\alpha}[(\mathfrak{M}_d - z)^{-1}d(k)]_{\alpha}h_{\alpha}$$

for $f \in \mathfrak{S}_m$, $k \in \mathfrak{K}_A$, and $z \in \text{res}A_0 = \text{res}L \setminus \{z_1\}$. The γ -field γ_{Γ} and the Weyl function M_{Γ} corresponding to $(\mathbb{C}^d, \Gamma_0, \Gamma_1)$ are described by

$$\gamma_{\Gamma}(z)\mathbb{C}^d = \mathfrak{N}_z(A_{\max}) = \left\{ \sum_{\sigma} c_{\sigma} F_{\sigma}(z) \mid c_{\sigma} \in \mathbb{C} \right\}, \quad F_{\sigma}(z) := \frac{g_{\sigma}(z)}{(z - z_1)^m}$$

and

$$M_{\Gamma}(z) = q(z) + r(z) \quad \text{on } \mathbb{C}^d$$

for $z \in \text{res}A_0$. The Krein Q -function q of L_{\min} is defined by

$$q(z) = ([q(z)]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [q(z)]_{\sigma\sigma'} := (z - z_1) \langle \varphi_{\sigma}, (L - z)^{-1} h_{\sigma', m+1} \rangle$$

for $z \in \text{res}L$, and the generalized Nevanlinna function r is defined by

$$r(z) = ([r(z)]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [r(z)]_{\sigma\sigma'} := - \sum_j \frac{[\mathcal{G}_A]_{\sigma m, \sigma' j}}{(z - z_1)^{m-j+1}}$$

for $z \in \mathbb{C} \setminus \{z_1\}$.

The compressed resolvent of A_{Θ} is represented in the generalized sense according to

$$P_{\mathfrak{S}_m}(A_{\Theta} - z)^{-1} \Big|_{\mathfrak{S}_m} = (L - z)^{-1} + \sum_{\sigma} [(\Theta - M_{\Gamma}(z))^{-1} \langle \varphi, (L - z)^{-1} \cdot \rangle]_{\sigma} (L - z)^{-1} h_{\sigma m} \quad (2.6)$$

for $z \in \text{res}A_0 \cap \text{res}A_{\Theta}$. As expected, in the A-model with equal model parameters the spectral properties of singular rank- d perturbations of class \mathfrak{S}_{-4} or higher are described by a generalized Nevanlinna function M_{Γ} .

3. Extensions which are linear relations

Let $j_* \in J$; then

$$\mathfrak{H}_m \cap \mathfrak{H}_{-m-2+2j_*} = \mathfrak{H}_m$$

while

$$\mathfrak{K}_A \cap \mathfrak{H}_{-m-2+2j_*} = \text{span}\{h_{\sigma j} \mid (\sigma, j) \in \mathcal{S} \times \{j_*, \dots, m\}\}$$

is a $d(m - j_* + 1)$ -dimensional linear space. Choosing $j_* = m$ we therefore construct a d -dimensional subspace \mathfrak{K}_A^{\min} of \mathfrak{K}_A , which is minimal in the sense that $\mathfrak{K}_A \cap \mathfrak{H}_{m-1} = \{0\}$. Let

$$\mathcal{H}_A^{\min} := (\mathfrak{H}_m \dot{+} \mathfrak{K}_A^{\min}, [\cdot, \cdot]_A).$$

That is, \mathcal{H}_A^{\min} is a subspace of \mathcal{H}_A equipped with an indefinite metric

$$[f + h_m(c), f' + h_m(c')]_A = \langle f, f' \rangle_m + \langle c, \mathcal{G}_A^{\min} c' \rangle_{\mathbb{C}^d}$$

for $f, f' \in \mathfrak{H}_m$ and $c, c' \in \mathbb{C}^d$. The matrix

$$\mathcal{G}_A^{\min} = ([\mathcal{G}_A^{\min}]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [\mathcal{G}_A^{\min}]_{\sigma\sigma'} := [\mathcal{G}_A]_{\sigma m, \sigma' m}$$

where, as previously, \mathcal{G}_A is the Gram matrix of the A -model; *i.e.* it is invertible and Hermitian. The matrix \mathcal{G}_A^{\min} is Hermitian, and the space \mathcal{H}_A^{\min} is a Hilbert space iff an Hermitian \mathcal{G}_A^{\min} is positive definite. In this case \mathcal{H}_A^{\min} becomes a subspace of the positive subspace of the Pontryagin space \mathcal{H}_A .

LEMMA 3.1. *Let \mathcal{H}_A^\perp denote the orthogonal complement in \mathcal{H}_A of \mathcal{H}_A^{\min} . Then:*

(i) \mathcal{H}_A^\perp is a subset of \mathfrak{K}_A given by

$$\mathcal{H}_A^\perp = \{k \in \mathfrak{K}_A \mid [\mathcal{G}_A d(k)]_m = 0\}.$$

(ii) Assume that

$$[\mathcal{G}_A]_{\sigma, m-1; \sigma' m} = [\mathcal{G}_A]_{\sigma m; \sigma', m-1}, \quad \sigma, \sigma' \in \mathcal{S} \tag{3.1}$$

if $m \geq 2$. Then

$$(A'_{\max} - A_{\max})\mathfrak{K}_A^{\min} \subseteq \mathcal{H}_A^\perp.$$

Recall that $A'_{\max} = A_{\max}$ if $m = 1$.

Proof. (i) \mathcal{H}_A^\perp is the set of $g + k \in \mathfrak{H}_m \dot{+} \mathfrak{K}_A$ such that $(\forall f \in \mathfrak{H}_m) (\forall c \in \mathbb{C}^d)$

$$0 = \langle f, g \rangle_m + \langle \eta(c), \mathcal{G}_A d(k) \rangle_{\mathbb{C}^{md}} = \langle f, g \rangle_m + \langle c, [\mathcal{G}_A d(k)]_m \rangle_{\mathbb{C}^d};$$

hence such that $g = 0$ and $[\mathcal{G}_A d(k)]_m = 0$.

(ii) We have $(\forall c \in \mathbb{C}^d)$

$$(A'_{\max} - A_{\max})h_m(c) = \tilde{k}'' \in \mathfrak{K}_A, \quad d(\tilde{k}'') = (\mathcal{G}_A^{-1} \mathcal{G}_{\mathfrak{M}}^* - \mathfrak{M}_d)\eta(c).$$

Then $(\forall \sigma \in \mathcal{S})$

$$\begin{aligned} [\mathcal{G}_A d(\tilde{k}'')]_{\sigma m} &= [(\mathcal{G}_{\mathfrak{M}}^* - \mathcal{G}_{\mathfrak{M}})\eta(c)]_{\sigma m} \\ &= \sum_{\sigma'} ([\mathcal{G}_A]_{\sigma, m-1; \sigma' m} - [\mathcal{G}_A]_{\sigma m; \sigma', m-1}) c_{\sigma'} . \end{aligned}$$

By hypothesis one therefore sees that $\tilde{k}'' \in \mathcal{H}_A^\perp$. \square

Define a linear relation B_{\max} in a (generally) Pontryagin space \mathcal{H}_A by

$$B_{\max} := A_{\max} \upharpoonright_{\text{dom} A_{\max} \cap \mathcal{H}_A^{\min}} \hat{+} (\{0\} \times \mathcal{H}_A^\perp)$$

(the componentwise sum), where

$$A_{\max} \upharpoonright_{\text{dom} A_{\max} \cap \mathcal{H}_A^{\min}} = A_{\max} \cap (\mathcal{H}_A^{\min} \times \mathcal{H}_A)$$

is the domain restriction to $\text{dom} A_{\max} \cap \mathcal{H}_A^{\min}$ of A_{\max} . Let also

$$B_{\min} := B_{\max}^*$$

be the adjoint in \mathcal{H}_A of B_{\max} .

For $m = 1$ we have $\mathcal{H}_A^{\min} = \mathcal{H}_A$ and $\mathcal{H}_A^\perp = \{0\}$, corresponding to $\mathfrak{K}_A^{\min} = \mathfrak{K}_A$. In this case $B_{\max} = A_{\max}$ and $B_{\min} = A_{\min}$ are operators. But for $m \geq 2$, B_{\max} has a nontrivial multivalued part $\text{mul} B_{\max} = \mathcal{H}_A^\perp$. The multivalued part of B_{\min} is also \mathcal{H}_A^\perp , which is seen from $\text{mul} B_{\min} = (\text{dom} B_{\max})^\perp$ and using that \mathfrak{H}_{m+2} is dense in \mathfrak{H}_m . We have, moreover, the next lemma.

LEMMA 3.2. *Assume (3.1) if $m \geq 2$. Then*

$$B_{\max} = A'_{\max} \upharpoonright_{\text{dom} A_{\max} \cap \mathcal{H}_A^{\min}} \hat{+} (\{0\} \times \mathcal{H}_A^\perp)$$

and

$$\begin{aligned} B_{\min} &= A_{\min} \upharpoonright_{\text{dom} A_{\min} \cap \mathcal{H}_A^{\min}} \hat{+} (\{0\} \times \mathcal{H}_A^\perp) \\ &= A'_{\min} \upharpoonright_{\text{dom} A_{\min} \cap \mathcal{H}_A^{\min}} \hat{+} (\{0\} \times \mathcal{H}_A^\perp) . \end{aligned}$$

Moreover, B_{\min} is a closed symmetric linear relation in \mathcal{H}_A , whose adjoint in \mathcal{H}_A is the linear relation $B_{\min}^* = B_{\max}$.

Proof. For $m = 1$ the statements of the lemma follow from Lemma 2.1, so in what follows we let $m \geq 2$.

The representation of B_{\max} , as stated, is due to Lemma 3.1. The adjoint of B_{\max} is given by (recall e.g. [19, Lemma 2.6])

$$B_{\min} = (A_{\max} \upharpoonright_{\text{dom} A_{\max} \cap \mathcal{H}_A^{\min}})^* \cap (\{0\} \times \mathcal{H}_A^\perp)^*$$

with

$$(\{0\} \times \mathcal{H}_A^\perp)^* = \mathcal{H}_A^{\min} \times \mathcal{H}_A.$$

Because A_{\max} and \mathcal{H}_A^{\min} are closed, by the same argument we also get that

$$\begin{aligned} (A_{\max} \upharpoonright_{\text{dom } A_{\max} \cap \mathcal{H}_A^{\min}})^* &= [A_{\max} \cap (\mathcal{H}_A^{\min} \times \mathcal{H}_A)]^* \\ &= \overline{A_{\min} \widehat{+} (\{0\} \times \mathcal{H}_A^\perp)}. \end{aligned}$$

Because

$$A_{\min}^* \widehat{+} (\{0\} \times \mathcal{H}_A^\perp)^* = A_{\max} \widehat{+} (\mathcal{H}_A^{\min} \times \mathcal{H}_A) = \mathcal{H}_A^2$$

is a closed linear relation, we have by [19, Lemma 2.10] that

$$\overline{A_{\min} \widehat{+} (\{0\} \times \mathcal{H}_A^\perp)} = A_{\min} \widehat{+} (\{0\} \times \mathcal{H}_A^\perp)$$

is also closed. Combining all together we deduce the first representation of B_{\min} as stated in the lemma. By using this representation and noting that $A_{\min} \subseteq A'_{\max}$ and $A'_{\min} \subseteq A_{\max}$ (Lemma 2.1), we deduce also the second formula for B_{\min} by applying Lemma 3.1. The computation of the adjoint B_{\min}^* uses the same arguments as that of B_{\max}^* , and one concludes that B_{\min} is a closed symmetric linear relation. \square

The boundary value space of B_{\min} is characterized by the next theorem.

THEOREM 3.3. *Assume (3.1) if $m \geq 2$, and let \mathcal{G}_A^{\min} be positive definite. Define the operator $\Gamma' := (\Gamma'_0, \Gamma'_1): B_{\max} \rightarrow \mathbb{C}^{2d}$ by*

$$\Gamma'_0 \widehat{f} := c, \quad \Gamma'_1 \widehat{f} := \langle \varphi, f^\# \rangle - \mathcal{G}_A^{\min} \chi$$

for $\widehat{f} = (f, f') \in B_{\max}$; that is

$$\begin{aligned} f &= f^\# + h_{m+1}(c) + h_m(\chi), \quad f^\# \in \mathfrak{H}_{m+2}, \quad c, \chi \in \mathbb{C}^d, \\ f' &= L_m f^\# + z_1 h_{m+1}(c) + \widetilde{k} + k_\perp, \quad \widetilde{k} \in \mathfrak{K}_A, \quad k_\perp \in \mathcal{H}_A^\perp, \end{aligned}$$

$$d(\widetilde{k}) = \mathfrak{M}_d \eta(\chi) + \eta(c).$$

Then $(\mathbb{C}^d, \Gamma'_0, \Gamma'_1)$ is a boundary triple for B_{\max} . The corresponding γ -field $\gamma_{\Gamma'}$ and the Weyl function $M_{\Gamma'}$ are bounded analytic operator functions given by

$$\gamma_{\Gamma'}(z) \mathbb{C}^d = \mathfrak{N}_z(B_{\max}) = \{(L - z)^{-1} h_m(c) + h_m(\chi) \mid \chi = (z - \widehat{\Delta})^{-1} c; c \in \mathbb{C}^d\},$$

$$\widehat{\Delta} := (\mathcal{G}_A^{\min})^{-1} \Delta \in [\mathbb{C}^d], \quad \Delta = (\Delta_{\sigma\sigma'}) = \Delta^* \in [\mathbb{C}^d],$$

$$\Delta_{\sigma\sigma'} := [\mathcal{G}_m]_{\sigma m, \sigma' m} = z_1 [\mathcal{G}_A^{\min}]_{\sigma\sigma'} + 1_{\mathbb{N}_{\geq 2}}(m) [\mathcal{G}_A]_{\sigma, m-1; \sigma' m}$$

and

$$M_{\Gamma'}(z) = q(z) + \widehat{r}(z), \quad \widehat{r}(z) := \mathcal{G}_A^{\min} (\widehat{\Delta} - z)^{-1}$$

for $z \in \text{res } L \cap \text{res } \widehat{\Delta}$. Moreover, $M_{\Gamma'}$ is a uniformly strict Nevanlinna function.

Proof. Step 1. In this step we argue as in the proof of Lemma 3.2. Consider Γ' as a single-valued linear relation with $\text{dom}\Gamma' = B_{\max}$:

$$\Gamma' = \{(\widehat{f}, (\Gamma'_0 \widehat{f}, \Gamma'_1 \widehat{f})) \mid \widehat{f} \in B_{\max}\}.$$

Likewise, consider Γ as a single-valued linear relation with $\text{dom}\Gamma = A_{\max}$:

$$\Gamma = \{(\widehat{f}, (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f})) \mid \widehat{f} \in A_{\max}\}.$$

Then by definition

$$\Gamma' = (\Gamma \cap \mathfrak{M}) \widehat{\mp} \mathfrak{N}, \quad \mathfrak{M} := (\mathcal{H}_A^{\min} \times \mathcal{H}_A) \times \mathbb{C}^{2d}, \quad \mathfrak{N} := (\{0\} \times \mathcal{H}_A^\perp) \times \{0\}.$$

Then the Krein space adjoint of Γ' is given by

$$(\Gamma')^{[*]} = (\Gamma \cap \mathfrak{M})^{[*]} \cap \mathfrak{N}^{[*]}, \quad \mathfrak{N}^{[*]} = \mathbb{C}^{2d} \times (\mathcal{H}_A^{\min} \times \mathcal{H}_A) = \mathfrak{M}^{-1}.$$

Because \mathfrak{M} is a closed linear relation, and so is Γ by Lemma 2.1(i), the Krein space adjoint of $\Gamma \cap \mathfrak{M}$ is given by

$$(\Gamma \cap \mathfrak{M})^{[*]} = \overline{\Gamma^{[*]} \widehat{\mp} \mathfrak{M}^{[*]}}, \quad \mathfrak{M}^{[*]} = \{0\} \times (\{0\} \times \mathcal{H}_A^\perp) = \mathfrak{N}^{-1}.$$

Because $\Gamma \widehat{\mp} \mathfrak{M} = \mathcal{H}_A^2 \times \mathbb{C}^{2d}$, it follows that

$$(\Gamma \cap \mathfrak{M})^{[*]} = \Gamma^{[*]} \widehat{\mp} \mathfrak{N}^{-1} = [(\Gamma^{[*]})^{-1} \widehat{\mp} \mathfrak{N}]^{-1}$$

and therefore

$$(\Gamma')^{[*]} = [(\Gamma^{[*]})^{-1} \widehat{\mp} \mathfrak{N}]^{-1} \cap \mathfrak{M}^{-1} = \{[(\Gamma^{[*]})^{-1} \cap \mathfrak{M}] \widehat{\mp} \mathfrak{N}\}^{-1}.$$

By applying Lemma 2.1(i) and Lemma 3.1(ii), this leads to $(\Gamma')^{[*]} = (\Gamma')^{-1}$.

Since Γ' is single-valued, unitary, and with closed domain, we conclude that Γ' is surjective, and then the triple $(\mathbb{C}^d, \Gamma'_0, \Gamma'_1)$ is a boundary triple for B_{\max} .

Step 2. We compute the eigenspace of B_{\max} . For $f \in \mathfrak{N}_z(B_{\max})$, $z \in \mathbb{C}$, we have

$$0 = (L - z)f^\# + (z_1 - z)h_{m+1}(c), \quad 0 = \widetilde{k} + k_\perp - zh_m(\chi).$$

Then, for $z \in \text{res}L$, the first equation leads to

$$f^\# = (z - z_1)(L - z)^{-1}h_{m+1}(c) = -h_{m+1}(c) + (L - z)^{-1}h_m(c)$$

The second equation implies that

$$0 = d(\widetilde{k}) + d(k_\perp) - z\eta(\chi) \quad \text{or else} \quad d(k_\perp) = (z - \mathfrak{M}_d)\eta(\chi) - \eta(c).$$

Because $k_\perp \in \mathcal{H}_A^\perp$, we have that $[\mathcal{G}_A d(k_\perp)]_m = 0$; hence

$$0 = \mathcal{G}_A^{\min}(z\chi - c) - [\mathcal{G}_{\mathfrak{M}}\eta(\chi)]_m, \quad [\mathcal{G}_{\mathfrak{M}}\eta(\chi)]_m = \Delta\chi.$$

Because by hypothesis an Hermitian \mathcal{G}_A^{\min} is positive definite, the latter shows that

$$0 = (z - \hat{\Delta})\chi - c \Rightarrow \chi = (z - \hat{\Delta})^{-1}c, \quad z \in \text{res } \hat{\Delta}.$$

Step 3. By definition $\gamma_{\Gamma'}(z)c = f \in \mathfrak{N}_z(B_{\max})$; thus by Step 2, we get $\gamma_{\Gamma'}(z)$ as claimed. Again by definition $M_{\Gamma'}(z)c = \Gamma'_1(f, zf)$, $f \in \mathfrak{N}_z(B_{\max})$; thus by Step 2, we get $M_{\Gamma'}(z)$ as stated in the lemma.

Step 4. Because q is the Weyl function corresponding to the boundary triple $(\mathbb{C}^d, \hat{\Gamma}_0, \hat{\Gamma}_1)$ for the adjoint in \mathfrak{H}_m of L_{\min} , where ([23, Corollary 7.4])

$$\hat{\Gamma}_0(f^\# + h_{m+1}(c)) := c, \quad \hat{\Gamma}_1(f^\# + h_{m+1}(c)) := \langle \varphi, f^\# \rangle,$$

we have by *e.g.* [15, Theorem 1.4] that q is a uniformly strict Nevanlinna function.

By hypothesis imposed on \mathcal{G}_A , the matrix Δ is Hermitian, so the matrix function \hat{r} is symmetric with respect to the real axis, $\hat{r}(z)^* = \hat{r}(\bar{z})$, $z \in \text{res } \hat{\Delta}$. We prove that $\text{res } \hat{\Delta} \supseteq \mathbb{C} \setminus \mathbb{R}$. Because \hat{r} is analytic on $\text{res } \hat{\Delta}$, and moreover the matrix

$$\frac{\Im \hat{r}(z)}{\Im z} = AB(z), \quad \Im z \neq 0,$$

$$A := (\mathcal{G}_A^{\min})^{-2} > 0, \quad B(z) := \hat{r}(z)^*(\mathcal{G}_A^{\min})^{-1}\hat{r}(z) > 0$$

is similar to the positive definite matrix $B(z)^{1/2}AB(z)^{1/2}$, this would imply that \hat{r} is a uniformly strict Nevanlinna function.

The spectrum of $\hat{\Delta}$ consists of $z \in \mathbb{C}$ such that the determinant $\det(\hat{\Delta} - z) = 0$. Because $\hat{\Delta}$ is the product of two Hermitian matrices, using their spectral decompositions we get that z solves $\det(Y - z) = 0$, where the matrix $Y := \Lambda^{-1}X$, Λ is the positive definite diagonal matrix with the eigenvalues of \mathcal{G}_A^{\min} on its diagonal, and X is an Hermitian matrix. Because $Y = \Lambda^{-1/2}Y'\Lambda^{1/2}$ is similar to an Hermitian matrix $Y' := \Lambda^{-1/2}X\Lambda^{-1/2}$, we get that z is an eigenvalue of Y' , and hence belongs to \mathbb{R} . Consequently, $\text{res } \hat{\Delta} \supseteq \mathbb{C} \setminus \mathbb{R}$ as claimed.

The sum $M_{\Gamma'}$ of two uniformly strict Nevanlinna functions q and \hat{r} is itself of the same class, as can be deduced from [5, Lemma 2.6] [6, Proposition 3.2], and this accomplishes the proof of the theorem. \square

Under assumptions of Theorem 3.3, consider Γ' as a (unitary) single-valued linear relation with $\text{dom } \Gamma' = B_{\max}$. According to [2, Theorem 4.8], if Γ' is minimal, *i.e.* if the closed linear span

$$\mathcal{H}_s := \overline{\text{span}}\{\mathfrak{N}_z(B_{\max}) \mid z \in \text{reg } B_{\min}\}$$

($\text{reg } B_{\min}$ is the regularity domain of B_{\min} ; see *e.g.* [1, Eq. (6.14)]) coincides with \mathcal{H}_A , then $M_{\Gamma'}$ must be a generalized Nevanlinna function with a generally nontrivial number κ of negative squares (where κ is equal to the rank of indefiniteness of the Pontryagin space \mathcal{H}_A). Recall that $\mathcal{H}_s = \mathcal{H}_A$ means also that a closed symmetric linear relation B_{\min} is simple. If, however, Γ' is not minimal, then $M_{\Gamma'}$ is a generalized Nevanlinna function with $\kappa' \leq \kappa$ negative squares. By Theorem 3.3 we have $\kappa' = 0$, and by the next proposition this corresponds to the fact that Γ' is not a minimal boundary relation for B_{\max} for at least $m \geq 2$, unless $\mathcal{H}_A^\perp = \{0\}$; if the latter holds then by our hypothesis on \mathcal{G}_A the space $\mathcal{H}_A = \mathcal{H}_A^{\min}$ is a Hilbert space (for all $m \geq 1$), and hence $\kappa = 0$.

THEOREM 3.4. *Under assumptions of Theorem 3.3, $\emptyset \neq \mathcal{H}_s \subseteq \mathcal{H}_A^{\min}$. Moreover, if the only solutions $f \in \mathfrak{H}_m$ and $\chi \in \mathbb{C}^d$ to*

$$(\forall z \in \mathbb{C} \setminus \mathbb{R}) \langle \varphi, (L - z)^{-1} f \rangle = \hat{r}(z)\chi \tag{3.2}$$

are $f = 0$ and $\chi = 0$, then $\mathcal{H}_s = \mathcal{H}_A^{\min}$.

Proof. First we prove the next lemma.

LEMMA 3.5. $(\forall k \in \mathfrak{K}_A) (\exists \chi \in \mathbb{C}^d) (\exists k_{\perp} \in \mathcal{H}_A^{\perp}) d(k) = \eta(\chi) + d(k_{\perp})$.

Proof. Because every $f \in \mathcal{H}_A$ is of the form $f = f' + k_{\perp}$, for some $f' \in \mathcal{H}_A^{\min}$ and $k_{\perp} \in \mathcal{H}_A^{\perp}$, we have that $f' = f'' + h_m(\chi)$, for some $f'' \in \mathfrak{H}_m$ and $\chi \in \mathbb{C}^d$. Choosing $f'' = 0$ the claim follows. \square

That \mathcal{H}_s is nonempty follows from the following lemma (recall that $\text{res } L \cap \text{res } \hat{\Delta} \supseteq \mathbb{C} \setminus \mathbb{R}$).

LEMMA 3.6. $\text{reg } B_{\min} \supseteq \text{res } L \cap \text{res } \hat{\Delta}$.

Proof. We show that, for $z \in \text{res } L \cap \text{res } \hat{\Delta}$, the eigenspace $\mathfrak{N}_z(B_{\min}) = \{0\}$ and the range $\text{ran}(B_{\min} - z)$ is closed, from which the statement of the lemma follows.

The linear relation $B_{\min} = \ker \Gamma'$ explicitly reads

$$\begin{aligned} B_{\min} = \{ & (f^{\#} + h_m(\chi), L_m f^{\#} + \tilde{k} + k_{\perp}) \mid f^{\#} \in \mathfrak{H}_{m+2}; \chi \in \mathbb{C}^d; \\ & k_{\perp} \in \mathcal{H}_A^{\perp}; \tilde{k} \in \mathfrak{K}_A; d(\tilde{k}) = \mathfrak{M}_d \eta(\chi); \langle \varphi, f^{\#} \rangle = \mathcal{G}_A^{\min} \chi \}. \end{aligned}$$

Therefore $f \in \mathfrak{N}_z(B_{\min})$ solves

$$0 = (L_m - z)f^{\#}, \quad 0 = (\hat{\Delta} - z)\chi, \quad \langle \varphi, f^{\#} \rangle = \mathcal{G}_A^{\min} \chi.$$

Since $z \in \text{res } L_m = \text{res } L$, this leads to $f = 0$.

By applying Lemma 3.5 $\tilde{k} = h_m(\hat{\Delta}\chi) + k'_{\perp}$, $k'_{\perp} \in \mathcal{H}_A^{\perp}$. Therefore the range

$$\begin{aligned} \text{ran}(B_{\min} - z) = \{ & (L_m - z)f^{\#} + h_m((\hat{\Delta} - z)\chi) + k_{\perp} \mid f^{\#} \in \mathfrak{H}_{m+2}; \chi \in \mathbb{C}^d; \\ & k_{\perp} \in \mathcal{H}_A^{\perp}; \langle \varphi, f^{\#} \rangle = \mathcal{G}_A^{\min} \chi \} \quad (z \in \mathbb{C}) \\ = \{ & (L_m - z)f^{\#} + h_m(\chi) + k_{\perp} \mid f^{\#} \in \mathfrak{H}_{m+2}; \chi \in \mathbb{C}^d; \\ & k_{\perp} \in \mathcal{H}_A^{\perp}; \langle \varphi, f^{\#} \rangle = \hat{r}(z)\chi \} \quad (z \in \text{res } \hat{\Delta}). \end{aligned}$$

On the other hand, the closure $\overline{\text{ran}}(B_{\min} - z)$, $\bar{z} \in \text{res } L \cap \text{res } \hat{\Delta}$, is the orthogonal complement in \mathcal{H}_A of $\mathfrak{N}_{\bar{z}}(B_{\max})$; hence

$$\begin{aligned} \overline{\text{ran}}(B_{\min} - z) = \{ & f + k \in \mathfrak{H}_m \dot{+} \mathfrak{K}_A \mid (\forall c \in \mathbb{C}^d) \\ & 0 = \langle f, (L - \bar{z})^{-1} h_m(c) \rangle_m + \langle d(k), \mathcal{G}_A \eta(\chi) \rangle_{\mathbb{C}^{md}}; \\ & \chi = (\bar{z} - \hat{\Delta})^{-1} c \}. \end{aligned}$$

Note that $\text{res } L \cap \text{res } \hat{\Delta} \supseteq \mathbb{C} \setminus \mathbb{R}$ implies that also $z \in \text{res } L \cap \text{res } \hat{\Delta}$.

We have

$$\begin{aligned} \langle f, (L - \bar{z})^{-1} h_m(c) \rangle_m &= \langle \langle \varphi, (L - z)^{-1} f \rangle, c \rangle_{\mathbb{C}^d}, \\ \langle d(k), \mathcal{G}_A \eta(\chi) \rangle_{\mathbb{C}^{md}} &= \langle (z - \hat{\Delta}^*)^{-1} [\mathcal{G}_A d(k)]_m, c \rangle_{\mathbb{C}^d} \\ &= - \langle \hat{r}(z) (\mathcal{G}_A^{\min})^{-1} [\mathcal{G}_A d(k)]_m, c \rangle_{\mathbb{C}^d}. \end{aligned}$$

Putting $f^\# := (L - z)^{-1} f \in \mathfrak{H}_{m+2}$ and applying Lemma 3.5, *i.e.*

$$\begin{aligned} d(k) &= \eta(\chi') + k'_\perp, \quad k'_\perp \in \mathcal{H}_A^\perp, \quad \chi' := (\mathcal{G}_A^{\min})^{-1} [\mathcal{G}_A d(k)]_m \\ &\Rightarrow [\mathcal{G}_A d(k)]_m = \mathcal{G}_A^{\min} \chi', \end{aligned}$$

we deduce that

$$\begin{aligned} \overline{\text{ran}}(B_{\min} - z) &= \{ (L_m - z) f^\# + h_m(\chi) + k_\perp \mid f^\# \in \mathfrak{H}_{m+2}; \chi \in \mathbb{C}^d; \\ &\quad k_\perp \in \mathcal{H}_A^\perp; \langle \varphi, f^\# \rangle = \hat{r}(z) \chi \} = \text{ran}(B_{\min} - z) \end{aligned}$$

for $z \in \text{res } L \cap \text{res } \hat{\Delta}$. We remark that the functional

$$\Phi(\cdot) := (\mathcal{G}_A^{\min})^{-1} [\mathcal{G}_A d(\cdot)]_m : \mathfrak{K}_A \rightarrow \mathbb{C}^d$$

is surjective, and that therefore $\chi' = \Phi(k)$ ranges over all \mathbb{C}^d whenever k ranges over all \mathfrak{K}_A . This accomplishes the proof of the lemma. \square

Because $\mathfrak{N}_z(B_{\max}) \subseteq \mathcal{H}_A^{\min}$, $z \in \mathbb{C}$, and because $\mathbb{C} \setminus \mathbb{R} \subseteq \text{res } L \cap \text{res } \hat{\Delta}$, it follows that

$$\mathring{\mathcal{H}}_s := \overline{\text{span}}\{ \mathfrak{N}_z(B_{\max}) \mid z \in \mathbb{C} \setminus \mathbb{R} \} \subseteq \mathcal{H}_s \subseteq \mathcal{H}_A^{\min}.$$

By the proof of Lemma 3.6, the orthogonal complement $\mathring{\mathcal{H}}_s^\perp$ in \mathcal{H}_A of $\mathring{\mathcal{H}}_s$ is given by

$$\mathring{\mathcal{H}}_s^\perp = \bigcap_{z \in \mathbb{C} \setminus \mathbb{R}} \text{ran}(B_{\min} - z) = X[\dot{+}] \mathcal{H}_A^\perp$$

where the subset $X \subseteq \mathcal{H}_A^{\min}$ is defined by

$$X := \{ f + h_m(\chi) \in \mathfrak{H}_m \dot{+} \mathfrak{K}_A^{\min} \mid (\forall z \in \mathbb{C} \setminus \mathbb{R}) \langle \varphi, (L - z)^{-1} f \rangle = \hat{r}(z) \chi \}$$

and $\dot{+}$ indicates the direct sum which is orthogonal with respect to the \mathcal{H}_A -metric $[\cdot, \cdot]_A$. If $X = \{0\}$, *i.e.* if (3.2) has the only solutions $f = 0$, $\chi = 0$, then $\mathring{\mathcal{H}}_s^\perp = \mathcal{H}_A^\perp$ implies $\mathring{\mathcal{H}}_s = \mathcal{H}_s = \mathcal{H}_A^{\min}$. \square

Assuming the hypotheses in Theorem 3.3, an extension $B_\Theta \in \text{Ext}(B_{\min})$ parametrized by a linear relation Θ in \mathbb{C}^d is defined by

$$B_\Theta := \{ \hat{f} \in B_{\max} \mid \Gamma' \hat{f} \in \Theta \}.$$

The Krein-Naimark resolvent formula for B_Θ is given by (*cf.* [15, Theorem 4.12])

$$(B_\Theta - z)^{-1} = (B_0 - z)^{-1} + \gamma_{\Gamma'}(z)(\Theta - M_{\Gamma'}(z))^{-1} \gamma_{\Gamma'}(\bar{z})^*, \quad z \in \text{res } B_0 \cap \text{res } B_\Theta$$

with $\gamma_{\Gamma'}(\bar{z})^* = \Gamma'_1(B_0 - z)^{-1}$. The self-adjoint extension $B_0 := \ker \Gamma'_0$ corresponds to the self-adjoint linear relation $\Theta = \{0\} \times \mathbb{C}^d$. The resolvent of B_0 is presented below.

PROPOSITION 3.7. *Assuming the hypotheses in Theorem 3.3 we have*

$$(B_0 - z)^{-1}(f + k) = (L_m - z)^{-1}f + h_m((\hat{\Delta} - z)^{-1}\Phi(k))$$

for $f \in \mathfrak{H}_m$, $k \in \mathfrak{K}_A$, and $z \in \text{res } B_0 = \text{res } L \cap \text{res } \hat{\Delta}$.

Proof. By applying Lemma 3.5

$$B_0 = \{(f^\# + h_m(\chi), L_m f^\# + h_m(\hat{\Delta}\chi) + k_\perp) \mid f^\# \in \mathfrak{H}_{m+2}; \chi \in \mathbb{C}^d; k_\perp \in \mathcal{H}_A^\perp\}.$$

Thus the eigenspace

$$\mathfrak{N}_z(B_0) = \mathfrak{N}_z(L_m) \dot{+} h_m(\mathfrak{N}_z(\hat{\Delta})), \quad z \in \mathbb{C}.$$

From here we see that the point spectrum

$$\sigma_p(B_0) = \sigma_p(L) \cup \sigma_p(\hat{\Delta}).$$

Then for $z \notin \sigma_p(B_0)$, the operator

$$(B_0 - z)^{-1} = \{(f + h_m(\chi) + k_\perp, (L_m - z)^{-1}f + h_m((\hat{\Delta} - z)^{-1}\chi)) \mid f \in \text{ran}(L_m - z); \chi \in \mathbb{C}^d; k_\perp \in \mathcal{H}_A^\perp\}$$

and it therefore follows that $\text{res } B_0 = \text{res } L \cap \text{res } \hat{\Delta}$. Putting $k := h_m(\chi) + k_\perp$ we have that $\chi = \Phi(k)$, and this leads to the resolvent formula as stated. \square

In view of Proposition 3.7, the compressed resolvent $P_{\mathfrak{H}_m}(B_\Theta - z)^{-1}|_{\mathfrak{H}_m}$ is given for $z \in \text{res } B_0 \cap \text{res } B_\Theta$ by the right hand side of (2.6), but where now M_Γ is replaced by $M_{\Gamma'}$.

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