

THE APS-INDEX AND THE SPECTRAL FLOW

KOEN VAN DEN DUNGEN AND LENNART RONGE*

(Communicated by F. Gesztesy)

Abstract. We study the Atiyah-Patodi-Singer (APS) index, and its equality to the spectral flow, in an abstract, functional analytic setting. More precisely, we consider a (suitably continuous or differentiable) family of self-adjoint Fredholm operators $A(t)$ on a Hilbert space, parametrised by t in a finite interval. We then consider two different operators, namely $D := \frac{d}{dt} + A$ (the abstract analogue of a Riemannian Dirac operator) and $D := \frac{d}{dt} - iA$ (the abstract analogue of a Lorentzian Dirac operator). The latter case is inspired by a recent index theorem by Bär and Strohmaier (Amer. J. Math. 141 (2019), 1421–1455) for a Lorentzian Dirac operator equipped with APS boundary conditions. In both cases, we prove that the Fredholm index of the operator D equipped with APS boundary conditions is equal to the spectral flow of the family $A(t)$.

1. Introduction

The relation between the index of a Riemannian Dirac operator and the spectral flow of a family of Dirac operators on hypersurfaces has been known since the work of Atiyah, Patodi, and Singer [2]. This relation has been generalised to a more abstract setting by Robbin and Salamon [18]. As index theory focuses primarily on *elliptic* operators, there has not been much work on index theory in the Lorentzian setting (the Lorentzian Dirac operator is *hyperbolic* instead of elliptic). Nevertheless, some work has been done in this direction. For instance, the index of the scattering operator for the Dirac equation (on a cylindrical Lorentzian manifold) has been studied in [9] (see also references therein). The Fredholm property and index of the (second-order) wave operator are studied in certain situations in [11].

In a recent paper [5], Bär and Strohmaier first considered the index of the Lorentzian Dirac operator with APS boundary conditions. They derived a Lorentzian version of the Atiyah-Patodi-Singer (APS) index theorem for globally hyperbolic spacetimes with future and past spacelike boundaries. One major aspect of their work is a Lorentzian version of the classical equality between the (APS) index and the spectral flow. The main purpose of this article is to recast this ‘APS index = spectral flow’ equality in a more abstract, functional analytic setting. For the sake of completeness and comparison, we will also discuss the ordinary Riemannian version of the Atiyah-Patodi-Singer

Mathematics subject classification (2020): 47A53, 58J20, 58J30.

Keywords and phrases: Atiyah-Patodi-Singer index, Fredholm pairs of projections, spectral flow.

The authors would like to thank the Hausdorff Center for Mathematics and the Bonn International Graduate School of Mathematics (BIGS-M) for their support.

* Corresponding author.

index in this abstract setting. Both in the Riemannian (elliptic) and in the Lorentzian (hyperbolic) case, we aim to prove the ‘APS index = spectral flow’ equality under minimal (functional analytic) assumptions, and we hope our work clarifies the differences and similarities between the two cases.

As the motivating example for this article, let us briefly recall the setting of [5]. Consider an even-dimensional, oriented, time-oriented Lorentzian spin manifold (X, g) . We will assume that (X, g) is *globally hyperbolic*, which implies [6, Theorem 1.1] that it is isometric to $(\mathbb{R} \times \Sigma, -N^2 dt^2 + g_t)$, where the *Cauchy hypersurface* Σ is a smooth manifold with a family of Riemannian metrics $\{g_t\}_{t \in \mathbb{R}}$, and the *lapse function* N is a smooth function $\mathbb{R} \times \Sigma \rightarrow (0, \infty)$. Furthermore, as in [5] we assume that the Cauchy hypersurface Σ is compact (for the noncompact case, see [7]).

Let ν be the past-directed unit normal vector field, and let $\beta = \gamma(\nu)$ be Clifford multiplication by ν . Since X is even-dimensional, the spinor bundle decomposes into spinors of positive and negative chirality. Identifying the positive and negative chirality spinors using β , the Lorentzian Dirac operator takes the form [21, Eq. (11)]

$$\not{D} = \begin{pmatrix} 0 & -\nabla_\nu + iA(t) - \frac{n}{2}H \\ -\nabla_\nu - iA(t) - \frac{n}{2}H & 0 \end{pmatrix},$$

where $A = \{A(t)\}_{t \in \mathbb{R}}$ is the family of Dirac operators on the Cauchy hypersurfaces $\{t\} \times \Sigma$, and H is the mean curvature of the hypersurfaces $\{t\} \times \Sigma$. If X is a metric product (i.e. $g = -dt^2 + g_0$, where the metric g_0 on Σ is independent of t), then we have $N \equiv 1$, $H \equiv 0$, and $A(t) = A_0$, and we obtain

$$\not{D} = \begin{pmatrix} 0 & \partial_t + iA_0 \\ \partial_t - iA_0 & 0 \end{pmatrix}.$$

In general, if X is not a metric product, we can use parallel transport along the curves $\mathbb{R} \rightarrow X$ given by $t \mapsto (t, x)$ for some $x \in \Sigma$, to show that we have [21, Proposition III.5]

$$\not{D} \simeq \begin{pmatrix} 0 & N^{-\frac{1}{2}} \partial_t N^{-\frac{1}{2}} + iB(t) \\ N^{-\frac{1}{2}} \partial_t N^{-\frac{1}{2}} - iB(t) & 0 \end{pmatrix},$$

where \simeq denotes unitary equivalence and $B(t)$ is obtained from $A(t)$ via the parallel transport isomorphism. The bottom left corner of the Dirac operator shall be denoted

$$D := -\nabla_\nu - iA(t) - \frac{n}{2}H \simeq N^{-\frac{1}{2}} \partial_t N^{-\frac{1}{2}} - iB(t).$$

We now restrict the globally hyperbolic spacetime $X = \mathbb{R} \times \Sigma$ to a finite time interval $[0, T]$. Thus we consider the globally hyperbolic spacetime $M := [0, T] \times \Sigma$, with past and future spacelike boundaries $\{0\} \times \Sigma$ and $\{T\} \times \Sigma$ (respectively). Since these spacelike boundaries are Riemannian manifolds, we can use the Dirac operators $A(0)$ and $A(T)$ to define Atiyah-Patodi-Singer (APS) boundary conditions (i.e., the domain is restricted to those functions f with $f(0)$ in the range of the negative spectral projection of $A(0)$ and $f(T)$ in the range of the positive spectral projection of $A(T)$). Thus equipping D with APS boundary conditions, Bär and Strohmaier [5] then prove

that the resulting operator D_{APS} is Fredholm, and that its index can be computed by the same formula as in the original (Riemannian) Atiyah-Patodi-Singer index theorem [1]. A crucial step in their proof is to relate this index to the spectral flow $\text{sf}(A)$ of the family $\{A(t)\}_{t \in [0, T]}$ of Dirac operators on the Cauchy hypersurfaces:

$$\text{ind}(D_{\text{APS}}) = \text{sf}(A). \tag{1}$$

Suppose now that our globally hyperbolic spacetime $M := [0, T] \times \Sigma$ is of product form near the boundary. Then in particular the lapse function N is equal to 1 near the boundary, so multiplication by $N^{\frac{1}{2}}$ preserves the APS boundary conditions. Hence we can consider the new operator

$$N^{\frac{1}{2}}D_{\text{APS}}N^{\frac{1}{2}} \simeq \partial_t - iN^{\frac{1}{2}}B(t)N^{\frac{1}{2}}$$

(where \simeq denotes unitary equivalence). Thus, writing $\tilde{A}(t) := N^{\frac{1}{2}}B(t)N^{\frac{1}{2}}$, we can summarise the above as follows: we wish to study the Fredholm index of an operator of the form $\partial_t - i\tilde{A}(t)$ with APS boundary conditions. The purpose of this article is to rederive Eq. (1) for such operators in a more general functional analytic setting:

- $A = \{A(t)\}_{t \in [0, T]}$ is a strongly continuously differentiable family of self-adjoint Fredholm operators on a Hilbert space \mathcal{H} with constant domain W ;
- D_{APS} is the closure of the operator

$$D := \frac{d}{dt} - iA$$

on $L^2([0, T], \mathcal{H})$, equipped with APS boundary conditions.

Furthermore, for the sake of completeness and comparison, we will also discuss the ‘Riemannian’ analogue, namely the operator $\frac{d}{dt} + A$ with APS boundary conditions.

Let us briefly summarise the contents of this article. First, in Section 2, some basic facts regarding strongly continuously differentiable families of operators will be derived for later use. In Section 3, we review the notion of spectral flow, following [17]. We prove in Theorem 3.5 that the spectral flow of a norm-continuous family $A = \{A(t)\}_{t \in [0, T]}$ is equal to the relative index of the pair $(P_{<0}(0), P_{<0}(T))$ of negative spectral projections of A at the endpoints, provided that $(P_{<0}(0), P_{<0}(t))$ is a Fredholm pair for each $t \in [0, T]$. This generalises a known result [15, Theorem 3.6] in the special case where $P_{<0}(0) - P_{<0}(t)$ is compact (cf. Remark 3.6).

In Section 4, we describe the abstract analogue of the *Riemannian* APS-index. We note that, on a Riemannian manifold $M = [0, T] \times \Sigma$ with the product metric $g = dt^2 + g_0$, the Dirac operator is of the form

$$\not{D} = \begin{pmatrix} 0 & -\partial_t + A_0 \\ \partial_t + A_0 & 0 \end{pmatrix},$$

where A_0 denotes the Dirac operator on the hypersurface Σ . We consider in Section 4 the more general setting where $A = \{A(t)\}_{t \in [0, T]}$ is a norm-continuous family of self-adjoint operators on a Hilbert space \mathcal{H} with constant domain W , where the inclusion

$W \hookrightarrow \mathcal{H}$ is compact. We then study the operator

$$D := \frac{d}{dt} + A$$

equipped with APS boundary conditions. We can extend A to a family \tilde{A} on the whole real line. We then recall from [2] the classical ‘index = spectral flow’ result:

$$\text{ind}\left(\frac{d}{dt} + \tilde{A}\right) = \text{sf}(\tilde{A}). \tag{2}$$

This equality has been rigorously proven by Robbin and Salamon [18] for a suitable differentiable family of operators $A = \{\tilde{A}(t)\}_{t \in \mathbb{R}}$. In fact, the assumption of differentiability is not necessary, and Eq. (2) remains valid for norm-continuous families (see [4, Theorem 2.1] and [22, Theorem 5.2]). We will prove (see Theorem 4.9) that the operator D_{APS} is Fredholm, and that we also have the equality

$$\text{ind}(D_{\text{APS}}) = \text{sf}(A).$$

The proof is based on relating the index of D_{APS} (on the interval $[0, T]$ with APS boundary conditions) to the index of the extension $\frac{d}{dt} + \tilde{A}$ (on the complete line \mathbb{R}). The main issue to overcome is that Eq. (2) is only valid for families with invertible endpoints, and we show that we may always perturb A to a family with invertible endpoints, without changing its spectral flow or the index of D_{APS} .

In Section 5, we finally describe the abstract analogue of the *Lorentzian* APS-index. In this case, we consider the operator

$$D := \frac{d}{dt} - iA$$

equipped with APS boundary conditions. Here we need to assume in addition that A is *strongly continuously differentiable*. The additional $-i$ before the family A leads to qualitatively very different behavior of the operator D . For instance, for the operator $\frac{d}{dt} + A$ on the real line, both the norm of $\frac{d}{dt}f$ and of Af can be estimated by the graph norm $\|f\|_{\frac{d}{dt}+A}$ (cf. [18]). For the operator $D = \frac{d}{dt} - iA$, however, the equation $Df = 0$ has solutions with arbitrarily large $\frac{d}{dt}f$ and Af . In fact, the equation has a unique solution for any initial value: indeed, the Cauchy problem corresponding to D is well-posed (see Theorem 5.4). Moreover, solutions to $Df = 0$ will not be square-integrable on \mathbb{R} , which necessitates restricting to a finite interval $[0, T]$ (and introducing boundary conditions).

In Section 5.1, we introduce the evolution operator Q , which describes solutions to the initial value problem

$$Df = 0, \quad f(s) = x.$$

The construction of this evolution operator, following [16, Ch. 5], requires the assumption that A is *strongly continuously differentiable*. We then use the evolution operator

in Section 5.2 to relate the index of D_{APS} to the (relative) index of a certain Fredholm pair of spectral projections at the endpoints of the interval, corresponding to the family of ‘evolved’ operators

$$\widehat{A}(t) = Q(0,t)A(t)Q(t,0).$$

We show in section 5.3 that \widehat{A} is again strongly continuously differentiable and therefore norm-continuous. In particular, we then know from Theorem 3.5 that the (relative) index of the pair of spectral projections of $\widehat{A}(0)$ and $\widehat{A}(T)$ is equal to the spectral flow of A . Thus we combine our results to prove the main theorem:

MAIN THEOREM. *If $(D|_{[0,t]})_{\text{APS}}$ is Fredholm for all $t \in [0, T]$, we have*

$$\text{ind}(D_{\text{APS}}) = \text{sf}(A).$$

Here $D|_{[0,t]}$ is the ‘restriction’ of D to the interval $[0, t]$. It follows from results in [5] that the Lorentzian Dirac operator studied there satisfies the hypothesis of our main theorem (see Example 5.10). The general idea and some parts of the proof of our main theorem are similar as in [5], while other parts are different. In particular, the use of Fredholm pairs and the aforementioned spectral projections allows for a much wider generalisation than a straightforward adaptation of the arguments of [5] would.

Finally, Section 5.4 will describe a counterexample which shows that Fredholmness of $(D|_{[0,t]})_{\text{APS}}$ is not a consequence of the other assumptions.

This article is largely based on the Master’s thesis by the second author ([19]), advised by Matthias Lesch and the first author. Several proofs which are only sketched in this article, can be found in more detail in [19].

The authors would like to thank Matthias Lesch for interesting discussions and for his helpful comments on this manuscript. The authors also thank the anonymous referees for their comments.

Notation

Let \mathcal{H} denote a separable, infinite-dimensional Hilbert space. For an operator T on \mathcal{H} and subspaces $X, Y \subset \mathcal{H}$ satisfying $X \subset \text{Dom } T$ and $\text{Ran } T \subset Y$, we denote by $T|_{X \rightarrow Y}$ the restriction of T to X with codomain Y .

Integrals and L^p -spaces of Banach-space-valued functions should be understood in the sense of Bochner integration (for details, see e.g. [12, Ch. 3]).

2. Families of operators

For this whole section, let X, Y and Z be Banach spaces, and let $a, b \in \mathbb{R}$. A family of operators $S: [a, b] \rightarrow B(X, Y)$ is called strongly continuous, if it is continuous with respect to the strong operator topology on $B(X, Y)$. It is called strongly continuously differentiable, if it is differentiable with respect to the strong operator topology and the derivative is strongly continuous. Explicitly, this means that there exists a strongly continuous family $S': [a, b] \rightarrow B(X, Y)$ such that for each $x \in X$ we have $\frac{d}{dt}(S(t)x) = S'(t)x$.

By the Banach-Steinhaus Theorem (or Uniform Boundedness Principle), strongly continuous families are uniformly bounded. As composition is continuous with respect to the strong topology when restricted to bounded subsets (in the operator norm), the composition of two strongly continuous families is again strongly continuous.

LEMMA 2.1. *Let $S: [a, b] \rightarrow B(X, Y)$ be strongly continuously differentiable. Then the following statements hold:*

1. S is norm-continuous.
2. If $S(t)$ is invertible for all $t \in [a, b]$, then the family

$$S^{-1}: [a, b] \rightarrow B(Y, X), \quad t \mapsto S(t)^{-1}$$

is strongly continuously differentiable with derivative $-S^{-1}S'S^{-1}$.

Proof. Norm-continuity at $t \in [a, b]$ is a consequence of the Banach-Steinhaus Theorem applied to

$$F := \left\{ \frac{1}{s-t}(S(s) - S(t)) \mid s \in [a, b] \setminus \{t\} \right\}.$$

As the inversion map is norm-continuous as well, also S^{-1} is norm-continuous (and in particular uniformly bounded). Let $t \in [a, b]$ and $h \in \mathbb{R}$ small enough such that $t + h \in [a, b]$. Then for $y \in Y$ we have (with the “little oh”-notation $o(h)$ referring to the norm in Y)

$$\begin{aligned} (S(t+h)^{-1} - S(t)^{-1})y &= S(t+h)^{-1}(S(t) - S(t+h))S(t)^{-1}y \\ &= -S(t+h)^{-1}(hS'(t)S(t)^{-1}y + o(h)) \\ &= -hS(t)^{-1}S'(t)S(t)^{-1}y + o(h), \end{aligned}$$

which proves the second statement. \square

PROPOSITION 2.2. *Let $X_0 \subseteq X$ be a dense subspace, and let $Y_0 \subseteq Y$ be a subspace of Y with a stronger norm that turns it into a Banach space. Consider a family of operators $S: [a, b] \rightarrow B(X, Y_0)$.*

1. *Suppose $S: [a, b] \rightarrow B(X, Y_0)$ is strongly continuous and $S: [a, b] \rightarrow B(X, Y)$ is strongly continuously differentiable. If $R: [a, b] \rightarrow B(Y, Z)$ is a strongly continuous family that restricts to a strongly differentiable family in $B(Y_0, Z)$, then $t \mapsto R(t)S(t)$ is strongly continuously differentiable in $B(X, Z)$, with derivative $R'(t)S(t) + R(t)S'(t)$.*
2. *Suppose that the restriction $S|_{X_0}: [a, b] \rightarrow B(X_0, Y)$ is strongly continuously differentiable, such that the derivative extends to a strongly continuous family $S': [a, b] \rightarrow B(X, Y_0)$. Then $S: [a, b] \rightarrow B(X, Y_0)$ is also strongly continuously differentiable with derivative S' .*

Proof. Since Banach-Steinhaus guarantees uniform boundedness of the differential quotients, composition can be treated like a continuous bilinear map and the calculation used to show the usual product rule can be applied verbatim to prove the first statement.

Regarding the second statement, for $x \in X_0$, we have

$$S(t)x = S(t_0)x + \int_{t_0}^t S'(r)x dr,$$

with the integral being taken in Y . Since the integrals in Y_0 and Y coincide and both sides are bounded linear functions of x , we get the same equality in Y_0 for any $x \in X$. \square

REMARK 2.3. The above proposition asserts in particular that pointwise compositions of strongly continuously differentiable families of operators are again strongly continuously differentiable. We also note that this implies the analogous result for evaluation instead of composition: if $f: [0, T] \rightarrow Y$ is a function, we can set $S: [0, T] \rightarrow B(\mathbb{C}, Y)$, $S(t)(1) = f(t)$. Then $R(t)f(t)$ has the same regularity properties as $R(t)S(t)$ in the strong topology.

3. Spectral flow

ASSUMPTION 3.1. Let \mathcal{H} be a separable Hilbert space, let $W \subseteq \mathcal{H}$ be a dense subspace, and let $\{A(t)\}_{t \in [0, T]}$ be a family of unbounded self-adjoint Fredholm operators on \mathcal{H} with constant domain W . We equip W with the graph norm of $A(0)$. We assume that the family A is a norm-continuous map from $[0, T]$ to $B(W, H)$.¹

The notion of spectral flow for a path of self-adjoint operators was first defined by Atiyah and Lusztig, and it appeared in the work of Atiyah, Patodi, and Singer [2, §7]. Heuristically, the spectral flow of the family A counts the number of eigenvalues of $A(t)$ (counted with multiplicities) crossing 0 as t varies from 0 to T , i.e. the number of negative eigenvalues becoming positive minus that of positive eigenvalues becoming negative. In this article we will follow the analytic definition of spectral flow given by Phillips in [17].

DEFINITION 3.2. Consider an interval $I \subset \mathbb{R}$, and let χ_I denote the characteristic function of I . For $t \in [0, T]$, consider the spectral projection of $A(t)$ and the corresponding spectral subspace given by

$$P_I(t) := \chi_I(A(t)), \quad \mathcal{H}_I(t) := \text{Ran}(P_I(t)).$$

For $a \in \mathbb{R}$, we will simply write

$$P_{<a}(t) := P_{(-\infty, a)}(t), \quad \mathcal{H}_{<a}(t) := \text{Ran}(P_{<a}(t)),$$

and similarly for $\geq a$.

¹We note here that W is complete (since $A(0)$ is closed), and it is then a consequence of the closed graph theorem that automatically $A(t) \in B(W, \mathcal{H})$ for each $t \in [0, T]$.

DEFINITION 3.3. ([17]) A partition

$$0 = t_0 < t_1 < \dots < t_N = T$$

together with numbers $a_n \in \mathbb{R}_{\geq 0}$ for $1 \leq n \leq N$ will be called a *flow partition* (for A), if for each n and $t \in [t_{n-1}, t_n]$ we have $a_n \notin \text{spec}(A(t))$ and $\mathcal{H}_{[0, a_n]}(t)$ is finite dimensional. For such a partition, the spectral flow is defined as

$$\text{sf}(A) = \sum_{n=1}^N \text{Dim}(\mathcal{H}_{[0, a_n]}(t_n)) - \text{Dim}(\mathcal{H}_{[0, a_n]}(t_{n-1})).$$

The spectral flow is well-defined, i.e. a flow partition exists and the spectral flow is independent of the choice of flow partition ([17]). We also note that the spectral flow is unchanged by conjugating with unitaries, as this does not change the dimensions of the spectral subspaces.

A pair (P, Q) of projections on \mathcal{H} is called a *Fredholm pair*, if the restricted operator $Q|_{\text{Ran}(P) \rightarrow \text{Ran}(Q)}$ is Fredholm. In this case the *(relative) index of (P, Q)* is defined to be the Fredholm index of $Q|_{\text{Ran}(P) \rightarrow \text{Ran}(Q)}$. If $P - Q$ is a compact operator, then (P, Q) is a Fredholm pair. For more details regarding the index of a pair of projections, we refer to [3]. We also quote the following result, which states that continuous families of Fredholm pairs have constant index:

LEMMA 3.4. ([15, Lemma 3.2]) *If $P, Q: [0, 1] \rightarrow B(\mathcal{H})$ are continuous paths of projections in some Hilbert space \mathcal{H} , such that $(P(t), Q(t))$ is a Fredholm pair for all $t \in [0, 1]$, then*

$$\text{ind}(P(0), Q(0)) = \text{ind}(P(1), Q(1)).$$

The following result relates the spectral flow of a family to the relative index of the spectral projections at the endpoints. Its proof combines arguments from [5, §4.2] (reformulated in terms of spectral projections) with Lemma 3.4.

THEOREM 3.5. *If $(P_{<0}(0), P_{<0}(t))$ is a Fredholm pair for all $t \in [0, T]$, we have*

$$\text{sf}(A) = \text{ind}(P_{<0}(0), P_{<0}(T)).$$

Proof. Let $(t_n), (a_n)$ be a flow partition for A . For any $a \in \mathbb{R}$, let $P_{<a}(t)_r$ denote the restriction

$$P_{<a}(t)_r := P_{<a}(t)|_{\mathcal{H}_{<0}(0) \rightarrow \mathcal{H}_{<a}(t)}.$$

Since $(P_{<0}(0), P_{<0}(t))$ is a Fredholm pair, we know that $P_{<0}(t)_r$ is Fredholm. Fix $n \leq N$. For $t \in [t_{n-1}, t_n]$, let

$$P_t := P_{<0}(t)|_{\mathcal{H}_{<a_n}(t) \rightarrow \mathcal{H}_{<0}(t)}$$

be the restriction of $P_{<0}(t)$, which is Fredholm since $\mathcal{H}_{[0, a_n]}(t)$ is finite-dimensional. We have

$$P_{<0}(t)_r = P_t P_{<a_n}(t)_r.$$

As $P_{<0}(t)_r$ and P_t are Fredholm, it follows that $P_{<a_n}(t)_r$ is Fredholm as well, and we have

$$\text{ind}(P_{<0}(t)_r) = \text{ind}(P_t) + \text{ind}(P_{<a_n}(t)_r) = \text{Dim}(\mathcal{H}_{[0,a_n]}^0(t)) + \text{ind}(P_{<a_n}(t)_r).$$

Since $a_n \notin \text{spec}(A(t))$ for $t \in [t_{n-1}, t_n]$, it follows from [14, Ch. 6, Theorem 5.12] that $P_{<a_n}(t)_r$ is continuous in t on $[t_{n-1}, t_n]$. By Lemma 3.4, $(P_{<0}(0), P_{<a_n}(t))$ has constant index for $t \in [t_{n-1}, t_n]$. Thus we have

$$\text{ind}(P_{<a_n}(t_n)_r) = \text{ind}(P_{<0}(0), P_{<a_n}(t_n)) = \text{ind}(P_{<0}(0), P_{<a_n}(t_{n-1})) = \text{ind}(P_{<a_n}(t_{n-1})_r).$$

Moreover, as $P_{<0}(0)_r$ is the identity on $\mathcal{H}_{(-\infty,0)}(0)$, it has index 0. Combining everything, we get:

$$\begin{aligned} \text{ind}(P_{<0}(0), P_{<0}(T)) &= \text{ind}(P_{<0}(T)_r) \\ &= \text{ind}(P_{<0}(T)_r) - \text{ind}(P_{<0}(0)_r) \\ &= \sum_{n=1}^N \text{ind}(P_{<0}(t_n)_r) - \text{ind}(P_{<0}(t_{n-1})_r) \\ &= \sum_{n=1}^N \text{Dim}(\mathcal{H}_{[0,a_n]}^0(t_n)) + \text{ind}(P_{<a_n}(t_n)_r) \\ &\quad - \text{Dim}(\mathcal{H}_{[0,a_n]}^0(t_{n-1})) - \text{ind}(P_{<a_n}(t_{n-1})_r) \\ &= \sum_{n=1}^N \text{Dim}(\mathcal{H}_{[0,a_n]}^0(t_n)) - \text{Dim}(\mathcal{H}_{[0,a_n]}^0(t_{n-1})) \\ &= \text{sf}(A). \quad \square \end{aligned}$$

REMARK 3.6. A similar theorem was proven in [15, Theorem 3.6]. There, the family A is only assumed to ‘Riesz continuous’ (instead of norm-continuous). On the other hand, [15, Theorem 3.6] makes the additional assumption that the difference $A(t) - A(0)$ is relatively compact (with respect to $A(0)$). The latter assumption ensures (by [15, Corollary 3.5]) that $P_{<0}(0) - P_{<0}(t)$ is compact, so in particular $(P_{<0}(0), P_{<0}(t))$ is a Fredholm pair for all $t \in [0, T]$. Thus, in the case of norm-continuous families, our Theorem 3.5 generalises [15, Theorem 3.6], since we do not require compactness of $P_{<0}(0) - P_{<0}(t)$.

4. The ‘Riemannian’ APS-index

In this section, we slightly strengthen Assumption 3.1 by assuming that $A(t)$ is not only Fredholm but in fact has compact resolvents. Thus throughout this section we consider the following setting.

ASSUMPTION 4.1. Let \mathcal{H} be a separable Hilbert space, let $W \subseteq \mathcal{H}$ be a dense subspace such that the inclusion is compact, and let $\{A(t)\}_{t \in [0, T]}$ be a family of unbounded self-adjoint operators on \mathcal{H} with constant domain W . We equip W with the graph norm of $A(0)$. We assume that the family $A: [0, T] \rightarrow B(W, \mathcal{H})$ is norm-continuous.

We continuously extend the family $\{A(t)\}_{t \in [0, T]}$ to a family $\{\tilde{A}(t)\}_{t \in \mathbb{R}}$ parametrised by the whole real line, defined by

$$\tilde{A}(t) := \begin{cases} A(0), & \text{if } t \leq 0, \\ A(t), & \text{if } 0 \leq t \leq T, \\ A(T), & \text{if } t \geq T. \end{cases}$$

We introduce the following spaces:

$$\mathcal{W} := L^2(\mathbb{R}, W) \cap H^1(\mathbb{R}, \mathcal{H}),$$

$$\mathcal{W}_{\text{APS}} := \{f \in L^2([0, T], W) \cap H^1([0, T], \mathcal{H}) : f(0) \in \mathcal{H}_{<0}(0), f(T) \in \mathcal{H}_{\geq 0}(T)\},$$

$$\mathcal{W}_{\text{APS}}^\dagger := \{f \in L^2([0, T], W) \cap H^1([0, T], \mathcal{H}) : f(0) \in \mathcal{H}_{\geq 0}(0), f(T) \in \mathcal{H}_{<0}(T)\}.$$

Here $H^1(\mathbb{R}) \subset L^2(\mathbb{R})$ denotes the standard first Sobolev space, and $H^1(\mathbb{R}, \mathcal{H}) \simeq H^1(\mathbb{R}) \otimes \mathcal{H}$ (as a completed tensor product).

We note that the evaluation $\text{ev}_t : H^1(\mathbb{R}, \mathcal{H}) \rightarrow \mathcal{H}$, $f \mapsto f(t)$, is well-defined (since elements in $H^1(\mathbb{R})$ are continuous).

DEFINITION 4.2. We consider the following operators:

- $\tilde{D} := \partial_t + \tilde{A}$ on the Hilbert space $L^2(\mathbb{R}, \mathcal{H})$ with initial domain $C_c^1(\mathbb{R}, W)$, and

$$\tilde{\mathcal{D}} := \begin{pmatrix} 0 & -\partial_t + \tilde{A} \\ \partial_t + \tilde{A} & 0 \end{pmatrix}$$

on the Hilbert space $L^2(\mathbb{R}, \mathcal{H})^{\oplus 2}$ with initial domain $C_c^1(\mathbb{R}, W)^{\oplus 2}$.

- $D_{\text{APS}} := \partial_t + A$ on the Hilbert space $L^2([0, T], \mathcal{H})$ with initial domain

$$\text{Dom } D_{\text{APS}} := \{f \in C^1([0, T], W) : f(0) \in \mathcal{H}_{<0}(0), f(T) \in \mathcal{H}_{\geq 0}(T)\},$$

and

$$\mathcal{D}_{\text{APS}} := \begin{pmatrix} 0 & -\partial_t + A \\ \partial_t + A & 0 \end{pmatrix}$$

on the Hilbert space $L^2([0, T], \mathcal{H})^{\oplus 2}$ with initial domain $\text{Dom } D_{\text{APS}} \oplus \text{Dom } D_{\text{APS}}^\dagger$, where

$$\text{Dom } D_{\text{APS}}^\dagger := \{f \in C^1([0, T], W) : f(0) \in \mathcal{H}_{\geq 0}(0), f(T) \in \mathcal{H}_{<0}(T)\}.$$

PROPOSITION 4.3.

1. The closure of the operator $\tilde{\mathcal{D}}$ is self-adjoint on the domain $\mathcal{W}^{\oplus 2}$, and for any $f \in C_0(\mathbb{R})$, the operators $f \cdot (\tilde{\mathcal{D}} \pm i)^{-1}$ on $L^2(\mathbb{R}, \mathcal{H})$ are compact.
2. The closure of the operator \mathcal{D}_{APS} is self-adjoint on the domain $\mathcal{W}_{\text{APS}} \oplus \mathcal{W}_{\text{APS}}^\dagger$, and the operators $(\mathcal{D}_{\text{APS}} \pm i)^{-1}$ on $L^2([0, T], \mathcal{H})$ are compact. In particular, \mathcal{D}_{APS} is Fredholm.

Proof. The self-adjointness of $\tilde{\mathcal{D}}$ on $\mathcal{W}^{\oplus 2}$ follows as in [22, Proposition 3.16]. Moreover, we know from [22, Proposition 4.1] (cf. the proof of [13, Theorem 6.7]) that $f \cdot (\tilde{\mathcal{D}} \pm i)^{-1}$ is compact for every $f \in C_0(\mathbb{R})$, which proves (1).

Next, we will prove the self-adjointness of \mathcal{D}_{APS} . Since A is norm-continuous, we can pick $0 < \varepsilon < \frac{1}{2}$ small enough such that

$$\sup_{t \in [0, \varepsilon]} \|(A(t) - A(0))(A(0) - i)^{-1}\| < \frac{1}{2}, \quad \sup_{t \in [T - \varepsilon, T]} \|(A(t) - A(T))(A(0) - i)^{-1}\| < \frac{1}{2}.$$

We consider a new norm-continuous family $A_L: [0, \infty) \rightarrow B(W, \mathcal{H})$ given by

$$A_L(t) := \begin{cases} A(t), & \text{if } 0 \leq t \leq \varepsilon, \\ A(\varepsilon), & \text{if } t \geq \varepsilon. \end{cases}$$

Consider the operators

$$\mathcal{D}_0 := \begin{pmatrix} 0 & -\partial_t + A(0) \\ \partial_t + A(0) & 0 \end{pmatrix}, \quad \mathcal{D}_L := \begin{pmatrix} 0 & -\partial_t + A_L \\ \partial_t + A_L & 0 \end{pmatrix},$$

on the Hilbert space $L^2([0, \infty), \mathcal{H})$ with domain $\mathcal{W}_L \oplus \mathcal{W}_L^\dagger$, where we introduce the spaces

$$\begin{aligned} \mathcal{W}_L &:= \{f \in L^2([0, \infty), W) \cap H^1([0, \infty), \mathcal{H}) : f(0) \in \mathcal{H}_{<0}(0)\}, \\ \mathcal{W}_L^\dagger &:= \{f \in L^2([0, \infty), W) \cap H^1([0, \infty), \mathcal{H}) : f(0) \in \mathcal{H}_{\geq 0}(0)\}. \end{aligned}$$

We recall that the operator \mathcal{D}_0 is self-adjoint (see [1, Proposition 2.12] or, for the more abstract setting, [8, Corollary 4.6] and [10, Proposition 4.11]). As in the proof of [22, Lemma 3.13], we can estimate

$$\begin{aligned} \|(\mathcal{D}_L - \mathcal{D}_0)(\mathcal{D}_0 - i)^{-1}\| &\leq \|(A_L - A(0))(A(0) - i)^{-1}\| \|(A(0) - i)(\mathcal{D}_0 - i)^{-1}\| \\ &\leq \sup_{t \in [0, \varepsilon]} \|(A(t) - A(0))(A(0) - i)^{-1}\| < \frac{1}{2}, \end{aligned}$$

where we have used that $\|(A(0) - i)(\mathcal{D}_0 - i)^{-1}\| \leq 1$. By the Kato-Rellich Theorem, it then follows that \mathcal{D}_L is also self-adjoint on the domain $\mathcal{W}_L \oplus \mathcal{W}_L^\dagger$. Similarly, the operator

$$\mathcal{D}_R := \begin{pmatrix} 0 & -\partial_t + A_R \\ \partial_t + A_R & 0 \end{pmatrix}, \quad A_R(t) := \begin{cases} A(T - \varepsilon), & \text{if } t \leq T - \varepsilon, \\ A(t), & \text{if } T - \varepsilon \leq t \leq T, \end{cases}$$

is self-adjoint on the domain $\mathcal{W}_R \oplus \mathcal{W}_R^\dagger$, where

$$\begin{aligned} \mathcal{W}_R &:= \{f \in L^2((-\infty, T], W) \cap H^1((-\infty, T], \mathcal{H}) : f(T) \in \mathcal{H}_{\geq 0}(T)\}, \\ \mathcal{W}_R^\dagger &:= \{f \in L^2((-\infty, T], W) \cap H^1((-\infty, T], \mathcal{H}) : f(T) \in \mathcal{H}_{<0}(T)\}. \end{aligned}$$

Now pick smooth functions $\chi_L, \chi_I, \chi_R: \mathbb{R} \rightarrow [0, 1]$ such that $\{\chi_L^2, \chi_I^2, \chi_R^2\}$ is a partition of unity subordinate to the open cover $\{(-\infty, \varepsilon), (0, T), (T - \varepsilon, \infty)\}$ of \mathbb{R} . For $\lambda > 0$, we define

$$R_\pm(\lambda) := \chi_L(\mathcal{D}_L \pm i\lambda)^{-1} \chi_L + \chi_I(\tilde{\mathcal{D}} \pm i\lambda)^{-1} \chi_I + \chi_R(\mathcal{D}_R \pm i\lambda)^{-1} \chi_R.$$

Since \mathcal{D}_{APS} agrees with \mathcal{D}_L on $[0, \varepsilon)$, agrees with $\tilde{\mathcal{D}}$ on $(0, T)$, and agrees with \mathcal{D}_R on $(T - \varepsilon, T]$, we note that $\text{Ran } R_{\pm}(\lambda) \subset \mathcal{W}_{\text{APS}} \oplus \mathcal{W}_{\text{APS}}^{\dagger}$, and we can compute

$$(\mathcal{D}_{\text{APS}} \pm i\lambda)R_{\pm}(\lambda) = \text{Id} + K_{\pm}(\lambda),$$

$$K_{\pm}(\lambda) := [\mathcal{D}_L, \chi_L](\mathcal{D}_L \pm i\lambda)^{-1}\chi_L + [\tilde{\mathcal{D}}, \chi_I](\tilde{\mathcal{D}} \pm i\lambda)^{-1}\chi_I + [\mathcal{D}_R, \chi_R](\mathcal{D}_R \pm i\lambda)^{-1}\chi_R.$$

By choosing λ large enough, we may ensure that $\|K_{\pm}(\lambda)\| < 1$, so that $\text{Id} + K_{\pm}(\lambda)$ is invertible, and then $R_{\pm}(\lambda)(\text{Id} + K_{\pm}(\lambda))^{-1}$ is a right inverse for $\mathcal{D}_{\text{APS}} \pm i\lambda$. Similarly, we can also construct a left inverse for $\mathcal{D}_{\text{APS}} \pm i\lambda$. Thus $\mathcal{D}_{\text{APS}} \pm i\lambda$ is invertible, which proves that \mathcal{D}_{APS} is self-adjoint.

Finally, we know from (1) that $\chi_I(\tilde{\mathcal{D}} \pm i\lambda)^{-1}$ is compact. Furthermore, the operator $\chi_L(\mathcal{D}_0 \pm i\lambda)^{-1}$ is compact by [10, Proposition 4.14], and since $\text{Dom } \mathcal{D}_L = \text{Dom } \mathcal{D}_0$ this implies that $\chi_L(\mathcal{D}_L \pm i\lambda)^{-1}$ is compact. Similarly, also $\chi_R(\mathcal{D}_R \pm i\lambda)^{-1}$ is compact. Hence also $R_{\pm}(\lambda)$ is compact, and therefore $(\mathcal{D}_{\text{APS}} \pm i)^{-1}$ is compact. This completes the proof of (2). \square

4.1. APS-index and spectral flow

We first consider the special case where the family A is invertible at the endpoints of the interval $[0, T]$. In this case, we recall the following equality between index and spectral flow on the real line.

THEOREM 4.4. ([4, Theorem 2.1]) *If $A(0)$ and $A(T)$ are invertible, then the operator \tilde{D} is Fredholm, and we have the equality*

$$\text{ind}(\tilde{D}) = \text{sf}(A).$$

PROPOSITION 4.5. *Assume that $A(0)$ and $A(T)$ are invertible. Then we have isomorphisms*

$$\text{Ker } D_{\text{APS}} \simeq \text{Ker } \tilde{D}, \quad \text{Ker } D_{\text{APS}}^* \simeq \text{Ker } \tilde{D}^*,$$

and consequently we have the equality

$$\text{ind}(D_{\text{APS}}) = \text{ind}(\tilde{D}).$$

Proof. The proof is an adaptation of the argument in [1, Proposition 3.11]. Let $\{\psi_{\lambda}(t)\}_{\lambda \in \text{spec}(A(t))}$ be an orthonormal basis of \mathcal{H} consisting of eigenvectors $\psi_{\lambda}(t)$ of $A(t)$ with eigenvalue λ (where the eigenvalues are counted with multiplicities). For any element $f \in \text{Ker } D_{\text{APS}}$, we can write $f(0) = \sum_{\lambda} \mu_{\lambda} \psi_{\lambda}(0)$, for some $\mu_{\lambda} \in \mathbb{C}$ (recall that the evaluation $\text{ev}_t: \text{Dom } D_{\text{APS}} \rightarrow \mathcal{H}$ is well-defined, since $\text{Dom } D_{\text{APS}} \subset H^1(\mathbb{R}, \mathcal{H})$). We will extend f to an element $\tilde{f} \in \text{Ker } \tilde{D}$, as follows. Solving $(\partial_t + \tilde{A})\tilde{f} = 0$ for $t < 0$ yields

$$\frac{\partial}{\partial t} \langle \psi_{\lambda}(0), \tilde{f}(t) \rangle = -\lambda \langle \psi_{\lambda}(0), \tilde{f}(t) \rangle,$$

which implies

$$\tilde{f}(t) = \sum_{\lambda < 0} e^{-\lambda t} \mu_{\lambda} \psi_{\lambda}(0), \quad t \leq 0.$$

Here we have used the APS boundary condition $f(0) \in \mathcal{H}_{<0}(0)$, which tells us that $\mu_\lambda = 0$ whenever $\lambda \geq 0$. Writing instead $f(T) = \sum_\lambda v_\lambda \psi_\lambda(T)$ and solving $(\partial_t + \tilde{A})\tilde{f} = 0$ for $t > T$, we similarly obtain

$$\tilde{f}(t) = \sum_{\lambda > 0} e^{-\lambda(t-T)} v_\lambda \psi_\lambda(T), \quad t \geq T,$$

where we have used that $A(T)$ is invertible, so that $\lambda \neq 0$. We can then define a map $\iota: \text{Ker } D_{\text{APS}} \rightarrow \text{Ker } \tilde{D}$ by defining

$$\iota(f)(t) := \begin{cases} \sum_{\lambda < 0} e^{-\lambda t} \mu_\lambda \psi_\lambda(0), & \text{if } t \leq 0, \\ f(t), & \text{if } 0 \leq t \leq T, \\ \sum_{\lambda > 0} e^{-\lambda(t-T)} v_\lambda \psi_\lambda(T), & \text{if } t \geq T. \end{cases}$$

This map ι is clearly injective. Conversely, given any $\xi \in \text{Ker } \tilde{D}$, the requirement that ξ is square-integrable ensures that ξ must have the above form on $(\infty, 0]$ and on $[T, \infty)$. By continuity, this implies that $\xi|_{[0, T]}$ satisfies the boundary conditions $\xi(0) \in \mathcal{H}_{<0}(0)$ and $\xi(T) \in \mathcal{H}_{>0}(T)$, and we conclude that $\xi = \iota(\xi|_{[0, T]})$. Thus we have shown that ι yields an isomorphism $\text{Ker } D_{\text{APS}} \xrightarrow{\cong} \text{Ker } \tilde{D}$. Similarly, we also obtain an isomorphism $\bar{\iota}: \text{Ker } D_{\text{APS}}^* \xrightarrow{\cong} \text{Ker } \tilde{D}^*$ given by

$$\bar{\iota}(f)(t) := \begin{cases} \sum_{\lambda > 0} e^{\lambda t} \mu_\lambda \psi_\lambda(0), & \text{if } t \leq 0, \\ f(t), & \text{if } 0 \leq t \leq T, \\ \sum_{\lambda < 0} e^{\lambda(t-T)} v_\lambda \psi_\lambda(T), & \text{if } t \geq T. \end{cases}$$

Since we know from Proposition 4.3 that D_{APS} is Fredholm, and from Theorem 4.4 that \tilde{D} is Fredholm, the final statement follows immediately. \square

Proposition 4.5 and Theorem 4.4 then immediately yield:

COROLLARY 4.6. *If $A(0)$ and $A(T)$ are invertible, then*

$$\text{ind}(D_{\text{APS}}) = \text{sf}(A).$$

Next, we will prove the equality $\text{ind}(D_{\text{APS}}) = \text{sf}(A)$ in general, by reducing to the special case with invertible endpoints, as follows.

DEFINITION 4.7. Consider a smooth function $\chi: \mathbb{R} \rightarrow [0, 1]$ such that $\chi \equiv 1$ near 0 and $\text{supp } \chi \subset (-\varepsilon, \varepsilon)$ for some $\varepsilon < \frac{1}{2}T$. We define a family of compact operators $\{K(t)\}_{t \in \mathbb{R}}$ on \mathcal{H} by

$$K(t) := \chi(t)P_0(A(0)) + \chi(T-t)P_0(A(T))$$

Here $P_0(A(t))$ denotes the projection onto the kernel of $A(t)$. We then obtain a new family $\{B(t)\}_{t \in [0, T]}$ of unbounded self-adjoint operators on \mathcal{H} with constant domain W , given by

$$B(t) := A(t) + K(t), \quad t \in [0, T].$$

We note that the family $\{B(t)\}_{t \in [0, T]}$ is again norm-continuous, and therefore satisfies Assumption 4.1. As above, we continuously extend $\{B(t)\}_{t \in [0, T]}$ to a family $\{\tilde{B}(t)\}_{t \in \mathbb{R}}$

on the real line. As in Definition 4.2 and Proposition 4.3, we then define the operators

$$\begin{aligned} \tilde{D}' &:= \partial_t + \tilde{B}, & \text{on } \text{Dom } \tilde{D}' &:= \mathscr{W}^{\oplus 2}, \\ D'_{\text{APS}} &:= \partial_t + B, & \text{on } \text{Dom } D'_{\text{APS}} &:= \mathscr{W}_{\text{APS}} \oplus \mathscr{W}_{\text{APS}}^\dagger. \end{aligned}$$

Let us make a few observations. First of all, the family $\{K(t)\}_{t \in [0, T]}$ is chosen such that the operators $B(0)$ and $B(T)$ are invertible. Second, we note that, in our conventions of both the spectral flow and the APS boundary conditions, zero belongs to the positive spectrum. Since the operators $K(0)$ and $K(T)$ move the kernels of $A(0)$ and $A(T)$ (respectively) into the strictly positive spectrum of $B(0)$ and $B(T)$ (respectively), we have $P_{\geq 0}(B(0)) = P_{\geq 0}(A(0))$ and $P_{\geq 0}(B(T)) = P_{\geq 0}(A(T))$. Consequently, we find that replacing A by B does not affect the APS boundary conditions, and we have the equality

$$\text{Dom } D'_{\text{APS}} = \text{Dom } D_{\text{APS}}.$$

LEMMA 4.8. *We have the equalities*

$$\text{sf}(B) = \text{sf}(A), \quad \text{ind}(D'_{\text{APS}}) = \text{ind}(D_{\text{APS}}).$$

Proof. We first prove the equality $\text{sf}(B) = \text{sf}(A)$. Since $K(t)$ is a family of compact operators, we can consider the straight-line homotopy $B_s := A + sK = \{A(t) + sK(t)\}_{t \in [0, 1]}$ for $s \in [0, 1]$. It then follows from [20, Cor. 3.4] that $\text{sf}(B) = \text{sf}(A)$, if the spectral flows $\text{sf}(\{B_s(0)\}_{s \in [0, 1]})$ and $\text{sf}(\{B_s(T)\}_{s \in [0, 1]})$ are both identically zero. That the latter condition is satisfied can be checked directly, using that the spectral projections $P_{\geq 0}(B_s(0))$ and $P_{\geq 0}(B_s(T))$ are constant.

Regarding the second equality, we recall from Proposition 4.3 that D_{APS} and D'_{APS} are Fredholm. We have already seen that D'_{APS} and D_{APS} have the same APS boundary conditions and therefore the same domain. Since the difference $D'_{\text{APS}} - D_{\text{APS}}$ is bounded and D_{APS} has compact resolvents by Proposition 4.3, we see that D'_{APS} is a relatively compact perturbation of D_{APS} , and therefore the index is the same. \square

THEOREM 4.9. *We have the equality*

$$\text{ind}(D_{\text{APS}}) = \text{sf}(A).$$

Proof. Combining the equalities from Lemma 4.8 with Corollary 4.6, we obtain the sequence of equalities

$$\text{ind}(D_{\text{APS}}) = \text{ind}(D'_{\text{APS}}) = \text{sf}(B) = \text{sf}(A). \quad \square$$

5. The ‘Lorentzian’ APS-index

In this section, we strengthen Assumption 3.1 by assuming that $A(t)$ is not only norm-continuous but in fact is strongly continuously differentiable. Thus throughout this section we consider the following setting.

ASSUMPTION 5.1. Let \mathcal{H} be a separable Hilbert space, let $W \subseteq \mathcal{H}$ be a dense subspace, and let $\{A(t)\}_{t \in [0, T]}$ be a family of unbounded self-adjoint Fredholm operators on \mathcal{H} with constant domain W . We equip W with the graph norm of $A(0)$. We assume that the family $A: [0, T] \rightarrow B(W, \mathcal{H})$ is strongly continuously differentiable.

DEFINITION 5.2. For $s < t \in [0, T]$, let $D|_{[s, t]}$ denote the closure in $L^2([s, t], \mathcal{H})$ of

$$\frac{d}{dt} - iA$$

with initial domain $C^1([s, t], \mathcal{H}) \cap C([s, t], W)$. Define $D := D|_{[0, T]}$.

5.1. The evolution operator

THEOREM 5.3. ([16, Ch. 5]) *There is a family of bounded operators $Q(t, s): \mathcal{H} \rightarrow \mathcal{H}$ for $s, t \in [0, T]$, satisfying the following conditions (for all $s, t, r \in [0, T]$):*

1. $Q(s, s) = \text{Id}$;
2. $Q(t, s)Q(s, r) = Q(t, r)$;
3. $Q(t, s)$ is an isometry (of \mathcal{H});
4. $Q(t, s)(W) \subseteq W$ and $Q(t, s): W \rightarrow W$ is bounded;
5. Q is strongly continuously differentiable in $B(W, \mathcal{H})$ with derivatives

$$\frac{\partial}{\partial t} Q(t, s) = iA(t)Q(t, s)$$

and

$$\frac{\partial}{\partial s} Q(t, s) = -Q(t, s)iA(s).$$

6. $Q(t, s)x$ (as a function of s and t) is continuous in \mathcal{H} for $x \in \mathcal{H}$ and continuous in W for $x \in W$.

Proof. Most of the statement is proven in [16, Ch. 5] for a more general situation (without assuming $A(t)$ to be self-adjoint). To be precise, [16, Ch. 5, Theorem 4.8] provides the operator $Q(t, s)$ for $t \geq s$, satisfying for all $t \geq s \geq r$ the conditions 1, 2, 4, and 6 (for 4 we note that the boundedness of $Q(t, s): W \rightarrow W$ follows from the inclusion $Q(t, s)(W) \subseteq W$ and the closed graph theorem), as well as the equalities $\frac{\partial}{\partial t}^+ Q(t, s) = iA(t)Q(t, s)$ and $\frac{\partial}{\partial s} Q(t, s) = -Q(t, s)iA(s)$. For $x \in W$, the calculation

$$\frac{\partial}{\partial t}^+ \|Q(t, s)x\|^2 = 2\text{Re}(\langle Q(t, s)x, iA(t)Q(t, s)x \rangle) = 0$$

together with $Q(s, s) = \text{Id}$ shows that $Q(t, s)$ is an isometry, so in fact 3 is also satisfied.

Similarly, we obtain the operator Q' associated to the family $-A(T - \cdot)$. Then the operator $Q(t, s) := Q'(T - t, T - s)$ satisfies the same conditions for all $r \geq s \geq t$.

Since both definitions agree at $t = s$, we get a strongly continuous family $Q(t, s)$ for all t and s . For $s \leq t$, we compute (using Proposition 2.2)

$$\frac{\partial^+}{\partial t} Q(s, t)Q(t, s) = -Q(s, t)iA(T - (T - t))Q(t, s) + Q(s, t)iA(t)Q(t, s) = 0.$$

Thus $Q(s, t)$ and $Q(t, s)$ are mutually inverse (as this holds at $t = s$), and we find that 2 is in fact satisfied for arbitrary s, t, r . Finally, as

$$\frac{\partial^\pm}{\partial t} Q(t, s) = \frac{\partial^\pm}{\partial r} Q(r, t) \Big|_{r=t} \quad Q(t, s) = iA(t)Q(t, s),$$

we get the t -derivatives in 5, and we note that $Q(t, s)$ is strongly continuously differentiable in $B(W, \mathcal{H})$ because $A(t)Q(t, s)$ is strongly continuous in $B(W, \mathcal{H})$. \square

We will refer to Q as the *evolution operator*. The unitary operator $Q(t, s)$ can be thought of as evolving the initial data at time s to the final data at time t , subject to the equation $Df = 0$. More precisely, the function $f(t) := Q(t, s)x$ is the unique solution to the equations

$$Df = 0, \quad f(s) = x,$$

When replacing $Df = 0$ with $Df = g$ for some $g \in L^2([0, T], \mathcal{H})$, the equations still have a unique solution:

THEOREM 5.4. (Well-posedness of the Cauchy problem) *The domain $\text{Dom}(D)$ is a subspace of $C([0, T], \mathcal{H})$ (with maximum norm) with bounded inclusion. For all $t \in [0, T]$ the map*

$$D \oplus \text{ev}_t : \text{Dom}(D) \rightarrow L^2([0, T], \mathcal{H}) \oplus \mathcal{H}$$

is an isomorphism, where $\text{ev}_t : C([0, T], \mathcal{H}) \rightarrow \mathcal{H}$ denotes evaluation at t .

Proof. For the first statement, let $f \in C^1([0, T], \mathcal{H}) \cap C([0, T], W)$. Using that $\text{Re}(\langle f(t), iA(t)f(t) \rangle) = 0$, we have

$$\begin{aligned} T\|f(t)\|^2 - \|f\|_{L^2}^2 &= \int_0^T (\|f(t)\|^2 - \|f(s)\|^2) ds = \int_0^T \int_s^t \frac{d}{dr} \|f(r)\|^2 dr ds \\ &= \int_0^T \int_s^t 2\text{Re} \left(\left\langle f(r), \frac{d}{dr} f(r) \right\rangle \right) dr ds \\ &= \int_0^T \int_s^t 2\text{Re}(\langle f(r), Df(r) \rangle) dr ds. \end{aligned}$$

This allows us to estimate

$$T\|f(t)\|^2 - \|f\|_{L^2}^2 \leq \int_0^T 2\|f\|_{L^2}\|Df\|_{L^2}ds \leq T\|f\|_D^2,$$

which ensures that the inclusion $C^1([0, T], \mathcal{H}) \cap C([0, T], W) \hookrightarrow C([0, T], \mathcal{H})$ extends to a bounded inclusion $\text{Dom}(D) \hookrightarrow C([0, T], \mathcal{H})$.

The second statement follows by checking that the map $F_s : L^2([0, T], \mathcal{H}) \oplus \mathcal{H} \rightarrow \text{Dom}(D)$, given for $g \in L^2([0, T], \mathcal{H})$ and $x \in \mathcal{H}$ by

$$F_s(g, x)(t) := Q(t, s)x + \int_s^t Q(t, r)g(r)dr,$$

is an inverse for $D \oplus \text{ev}_s$. We first note that F_s maps the dense subset $C^1([0, T], W) \oplus W$ to the initial domain $C^1([0, T], \mathcal{H}) \cap C([0, T], W)$ of D . Furthermore, an explicit computation shows that $F_s \circ (D \oplus \text{ev}_s)$ and $(D \oplus \text{ev}_s) \circ F_s$ are the identity on $C^1([0, T], \mathcal{H}) \cap C([0, T], W)$ and $C^1([0, T], W) \oplus W$ respectively. For (f, x) in the latter space, we can then estimate

$$\begin{aligned} \|F_s(f, x)\|_{L^2}^2 &\leq 2\|Q(\cdot, s)x\|_{L^2}^2 + 2\left\|\int_s^\cdot Q(\cdot, r)f(r)dr\right\|_{L^2}^2 \\ &\leq 2\|x\|_{L^2}^2 + 2\left\|\int_s^\cdot \|f(r)\|dr\right\|_{L^2}^2 \\ &\leq 2T\|x\|^2 + 2T\|f\|_{L^1}^2 \leq 2T\|x\|^2 + 2CT\|f\|_{L^2}^2, \end{aligned}$$

for some $C > 0$. It follows that

$$\|F_s(f, x)\|_D^2 = \|F_s(f, x)\|_{L^2}^2 + \|f\|_{L^2}^2 \leq (1 + 2T + 2CT)(\|f\|_{L^2}^2 + \|x\|^2).$$

Thus F_s extends to a continuous map $L^2([0, T], \mathcal{H}) \oplus \mathcal{H} \rightarrow \text{Dom}(D)$, whence the two compositions are the identity everywhere. \square

Using the above theorem, we can rewrite the evolution operator in a concise way that highlights its connection to the Cauchy problem:

$$Q(t, s)x = \text{ev}_t \circ (D \oplus \text{ev}_s)^{-1}(0, x). \tag{3}$$

5.2. The APS-index and spectral projections

In the following, we will use the splitting of \mathcal{H} in positive and negative spectral subspaces of $A(t)$, in order to define APS boundary conditions. For any $t \in [0, T]$, we consider (as before) the spectral projections

$$P_{<0}(t) := P_{(-\infty, 0)}(t), \quad P_{\geq 0}(t) := P_{[0, \infty)}(t) = \text{Id} - P_{<0}(t),$$

and the corresponding subspaces

$$\mathcal{H}_{<0}(t) := \text{Ran}(P_{<0}(t)), \quad \mathcal{H}_{\geq 0}(t) := \text{Ran}(P_{\geq 0}(t)).$$

DEFINITION 5.5. For $s < t \in [0, T]$, let $(D|_{[s,t]})_{\text{APS}}$ be the restriction of $D|_{[s,t]}$ to the domain

$$\text{Dom}((D|_{[s,t]})_{\text{APS}}) := \{f \in \text{Dom}(D) : f(s) \in \mathcal{H}_{<0}(s), f(t) \in \mathcal{H}_{\geq 0}(t)\}.$$

We will write

$$D_{\text{APS}} := (D|_{[0,T]})_{\text{APS}}.$$

We will relate the index of D_{APS} to the index of a pair of spectral projections. For this purpose, we consider the *evolved spectral projections* defined as

$$\widehat{P}_{<a}(t) := Q(0,t)P_{<a}(t)Q(t,0).$$

Let $\widehat{P}_{<a}(t)_r$ be the restriction of $\widehat{P}_{<a}(t)$ to $\mathcal{H}_{<0}(0)$ with codomain $Q(0,t)\mathcal{H}_{<a}(t)$:

$$\widehat{P}_{<a}(t)_r := \widehat{P}_{<a}(t)|_{\mathcal{H}_{<0}(0) \rightarrow Q(0,t)\mathcal{H}_{<a}(t)}.$$

We note that $\widehat{P}_{<a}(t)$ is the projection onto $Q(0,t)\mathcal{H}_{<a}(t)$, and that (by construction) $\widehat{P}_{<a}(t)_r$ is Fredholm with index k if and only if the pair $(P_{<0}(0), \widehat{P}_{<a}(t))$ is Fredholm with index k . The following result is partly based on the arguments from [5, §3].

THEOREM 5.6. D_{APS} and $\widehat{P}_{<0}(T)_r$ have isomorphic kernel and cokernel. In particular, D_{APS} is Fredholm with index k if and only if $(P_{<0}(0), \widehat{P}_{<0}(T))$ is a Fredholm pair with index k .

REMARK 5.7. By replacing A by $A|_{[0,t]}$, we obtain for any $t \in [0, T]$ that $(D|_{[0,t]})_{\text{APS}}$ is Fredholm with index k if and only if $(P_{<0}(0), \widehat{P}_{<0}(t))$ is Fredholm with index k .

Proof. We have

$$\begin{aligned} \text{Ker}(D_{\text{APS}}) &= \{f \in \text{Dom}(D) : Df = 0, f(0) \in \mathcal{H}_{<0}(0), f(T) \in \mathcal{H}_{\geq 0}(T)\} \\ &\cong \{f(0) \in \mathcal{H}_{<0}(0) : Q(T,0)f(0) \in \mathcal{H}_{\geq 0}(T)\} \\ &= \mathcal{H}_{<0}(0) \cap Q(0,T)\mathcal{H}_{\geq 0}(T) \\ &= \text{Ker}(\widehat{P}_{<0}(T)_r), \end{aligned}$$

where in the second line, we use that $Df = 0$ implies $f(t) = Q(t,0)f(0)$, so $f \mapsto f(0)$ is an isomorphism.

For $g \in L^2([0, T], \mathcal{H})$ define

$$E(g) := \text{ev}_T \circ (D \oplus \text{ev}_0)^{-1}(g, 0).$$

Note that

$$\text{ev}_T \circ (D \oplus \text{ev}_0)^{-1}(g, x) = E(g) + Q(T,0)x.$$

We will first show that $E : L^2([0, T], \mathcal{H}) \rightarrow \mathcal{H}$ is surjective. Thus, we need to show that functions in $\text{Dom}(D)$ that vanish at 0 can take any value at T . For $z \in \mathcal{H}$

choose $f \in \text{Dom}(D)$ with $f(T) = z$ (a possible choice is $f(t) = Q(t, T)z$) and let $\phi(t) := \frac{t}{T}$. Since multiplication with ϕ preserves $\text{Dom}(D)$, we have $\phi f \in \text{Dom}(D)$, with $\phi(0)f(0) = 0$ and $\phi(T)f(T) = z$. We get

$$E(D(\phi f)) = \text{ev}_T \circ (D \oplus \text{ev}_0)^{-1}(D(\phi f), 0) = \text{ev}_T(\phi f) = z.$$

As z was arbitrary, E is surjective.

To determine the cokernel of D_{APS} , we need to characterise its range. For $g \in L^2([0, T], \mathcal{H})$, we have the following chain of equivalences:

$$\begin{aligned} g \in \text{Ran}(D_{\text{APS}}) &\Leftrightarrow \exists f \in \text{Dom}(D) : f(0) \in \mathcal{H}_{<0}(0) \wedge f(T) \in \mathcal{H}_{\geq 0}(T) \wedge Df = g \\ &\Leftrightarrow \exists f(0) \in \mathcal{H}_{<0}(0) : \text{ev}_T(D \oplus \text{ev}_0)^{-1}(g, f(0)) \in \mathcal{H}_{\geq 0}(T) \\ &\Leftrightarrow \exists f(0) \in \mathcal{H}_{<0}(0) : \exists z \in \mathcal{H}_{\geq 0}(T) : E(g) + Q(T, 0)(f(0)) = z \\ &\Leftrightarrow \exists x \in Q(T, 0)\mathcal{H}_{<0}(0) : \exists z \in \mathcal{H}_{\geq 0}(T) : E(g) = z - x \\ &\Leftrightarrow E(g) \in Q(T, 0)\mathcal{H}_{<0}(0) + \mathcal{H}_{\geq 0}(T) \end{aligned}$$

Defining

$$V := Q(T, 0)\mathcal{H}_{<0}(0) + \mathcal{H}_{\geq 0}(T) = P_{<0}(T)Q(T, 0)\mathcal{H}_{<0}(0) + \mathcal{H}_{\geq 0}(T),$$

(with the latter sum being orthogonal), we get

$$\text{Ran}(D_{\text{APS}}) = \{g \in L^2([0, T], \mathcal{H}) : E(g) \in V\} = E^{-1}(V).$$

In particular, this also implies that $\text{Ker}(E) = E^{-1}(\{0\}) \subset \text{Ran}(D_{\text{APS}})$. By the surjectivity of E , we therefore obtain the isomorphism

$$L^2([0, T], \mathcal{H}) / \text{Ran}(D_{\text{APS}}) \cong \mathcal{H} / V.$$

We can now conclude

$$\begin{aligned} \text{Coker}(D_{\text{APS}}) &= L^2([0, T], \mathcal{H}) / \text{Ran}(D_{\text{APS}}) \\ &\cong \mathcal{H} / V \\ &\cong \mathcal{H}_{<0}(T) / (P_{<0}(T)Q(T, 0)\mathcal{H}_{<0}(0)) \\ &\cong (Q(0, T)\mathcal{H}_{<0}(T)) / (Q(0, T)P_{<0}(T)Q(T, 0)\mathcal{H}_{<0}(0)) \\ &= \text{Coker}(\widehat{P}_{<0}(T)_r). \quad \square \end{aligned}$$

5.3. APS-index and spectral flow

We recall that the strongly continuously differentiable family $\{A(t)\}_{t \in [0, T]}$ is norm-continuous by Lemma 2.1.1, so in particular the results from Section 3 apply. In order to combine Theorems 3.5 and 5.6, we need to consider a new ‘evolved’ family $\widehat{A}: [0, T] \rightarrow B(W, \mathcal{H})$ given by

$$\widehat{A}(t) := Q(0, t)A(t)Q(t, 0).$$

For every $t \in [0, T]$, $\widehat{A}(t)$ is self-adjoint and Fredholm, with domain W (as $Q(t, 0)^{-1}(W) = W$). As functional calculus is equivariant under conjugation with isometries, we find that the spectral projections of $\widehat{A}(t)$ correspond precisely to the evolved spectral projections from subsection 5.2:

$$\chi_{(-\infty, a)}(\widehat{A}(t)) = Q(0, t)\chi_{(-\infty, a)}(A(t))Q(t, 0) = \widehat{P}_{<a}(t).$$

Before we can apply Theorem 3.5 to \widehat{A} , we need to ensure that \widehat{A} is again norm-continuous, and we will prove that it is in fact strongly continuously differentiable in $B(W, \mathcal{H})$.

LEMMA 5.8. $\widehat{A}: [0, T] \rightarrow B(W, \mathcal{H})$ is strongly continuously differentiable with derivative

$$\widehat{A}'(t) = Q(0, t)A'(t)Q(t, 0).$$

Proof. Let $R(t) := (A(t) - i)^{-1}$ for $t \in [0, T]$. $\widehat{A}(t)$ is differentiable at t if and only if

$$\widehat{A}(t) - i = Q(0, t)(A(t) - i)Q(t, 0)$$

is. As $Q(t, 0)$ and $Q(0, t)$ are strongly continuously differentiable in $B(W, \mathcal{H})$ and $R(t)$ is strongly continuously differentiable in $B(\mathcal{H}, W)$, we get from Lemma 2.1.2 and Proposition 2.2.1 that

$$(\widehat{A}(t) - i)^{-1} = Q(0, t)R(t)Q(t, 0)$$

is strongly differentiable in $B(W, \mathcal{H})$. Its derivative is

$$\begin{aligned} & \frac{d}{dt}(\widehat{A}(t) - i)^{-1} \\ &= \frac{d}{dt}Q(0, t)R(t)Q(t, 0) \\ &= Q(0, t)R(t)iA'(t)Q(t, 0) - Q(0, t)R(t)A'(t)R(t)Q(t, 0) - Q(0, t)iA'(t)R(t)Q(t, 0) \\ &= -Q(0, t)R(t)A'(t)R(t)Q(t, 0). \end{aligned}$$

As this is strongly continuous in $B(\mathcal{H}, W)$, Proposition 2.2.2 implies that $(\widehat{A}(t) - i)^{-1}$ is strongly continuously differentiable in $B(\mathcal{H}, W)$. By Lemma 2.1.2, $\widehat{A}(t) - i$ and hence $\widehat{A}(t)$ are strongly continuously differentiable, with derivative

$$\begin{aligned} \widehat{A}'(t) &= \frac{d}{dt}((\widehat{A}(t) - i)^{-1})^{-1} \\ &= -(\widehat{A}(t) - i) \left(\frac{d}{dt}(\widehat{A}(t) - i)^{-1} \right) (\widehat{A}(t) - i) \\ &= Q(0, t)A'(t)Q(t, 0). \quad \square \end{aligned}$$

We now have all the pieces in place to prove our main result.

THEOREM 5.9. *If $(D|_{[0,t]})_{\text{APS}}$ is Fredholm for all $t \in [0, T]$, we have*

$$\text{ind}(D_{\text{APS}}) = \text{sf}(A).$$

Proof. From Lemma 5.8 we know that \widehat{A} satisfies Assumption 5.1. In particular, \widehat{A} is norm-continuous by Lemma 2.1.1, so we may apply Theorem 3.5. The spectral projections of \widehat{A} are given by

$$\chi_{(-\infty,0)}(\widehat{A}(t)) = \widehat{P}_{<0}(t).$$

Using Theorem 5.6, we know that $(\widehat{P}_{<0}(0), \widehat{P}_{<0}(t))$ is a Fredholm pair for all $t \in [0, T]$. Thus we obtain

$$\text{ind}(D_{\text{APS}}) \stackrel{5.6}{=} \text{ind}(P_{<0}(0), \widehat{P}_{<0}(T)) = \text{ind}(\widehat{P}_{<0}(0), \widehat{P}_{<0}(T)) \stackrel{3.5}{=} \text{sf}(\widehat{A}) = \text{sf}(A),$$

where in the last step we used that the spectral flow is invariant under unitary conjugation. \square

EXAMPLE 5.10. Consider the Lorentzian Dirac operator on a globally hyperbolic spacetime $M = \Sigma \times \mathbb{R}$, as studied in [5] (and as described in the Introduction). It is shown in [5, Lemma 2.6], using methods of Fourier integral operators, that the operator

$$Q_{--}(t, 0) := P_{<0}(t)Q(t, 0)|_{\mathcal{H}_{<0}(0) \rightarrow \mathcal{H}_{<0}(t)}$$

is Fredholm for each $t \in [0, T]$. Since $\widehat{P}_{<0}(t)_r = Q(0, t)Q_{--}(t, 0)$ and $Q(0, t)$ is an invertible map between the codomains, it then follows that $\widehat{P}_{<0}(t)_r$ is also Fredholm (and has the same index) for each $t \in [0, T]$. Using Theorems 5.6 and 5.9, we thus recover the equality² $\text{ind}(D_{\text{APS}}) = \text{sf}(A)$ from [5, §4.1].

It may be difficult to determine a priori whether $(D|_{[0,t]})_{\text{APS}}$ is Fredholm for all $t \in [0, T]$. The following result provides a sufficient condition.

PROPOSITION 5.11. *If $A'(t)$ is compact in $B(W, \mathcal{H})$ for all $t \in [0, T]$ (i.e., it is relatively compact with respect to $A(0)$), then D_{APS} is Fredholm and*

$$\text{ind}(D_{\text{APS}}) = \text{sf}(A).$$

Proof. By Lemma 5.8, \widehat{A}' is compact as well. This implies that $\widehat{A}(t) - A(0)$ is compact in $B(W, \mathcal{H})$ for every $t \in [0, T]$. From [15, Corollary 3.5], it follows that $\widehat{P}_{<0}(t) - P_{<0}(0)$ is compact, so $(P_{<0}(0), \widehat{P}_{<0}(t))$ is a Fredholm pair. By Theorems 5.6 and 5.9, we get the desired result. \square

REMARK 5.12. The counterexample in the next section shows that it is not sufficient to ask for relative compactness of $A(t) - A(0)$.

²In [5, §4.1], there is actually an additional summand on the right hand side coming from the kernel of $A(T)$, due to a slightly different choice of boundary conditions.

5.4. A counterexample with bounded perturbation

In this section an example is given to illustrate that D_{APS} will not always be Fredholm. There might be “infinite exchange” between the positive and the negative spectral subspace. This is possible, even if $A(t)$ has only discrete spectrum and its difference from $A(0)$ is bounded. The idea is to choose a bounded perturbation $A(t) = A(0) + B(t)$ such that the corresponding evolution operator $Q(T, 0)$ interchanges the positive and negative eigenspaces of $A(0)$ and $A(T)$. The first step is to show that such an exchange works in a two dimensional subspace, with suitable bounds on the derivative of the perturbation. These bounds will then allow us to pass to an infinite direct sum, in which all positive and negative eigenspaces are interchanged. This means that $\text{Ker}(D_{\text{APS}}) \cong \mathcal{H}_{<0}(0) \cap Q(0, T)\mathcal{H}_{\geq 0}(T)$ will be infinite-dimensional, whence D_{APS} is not Fredholm.

LEMMA 5.13. *There exists a positive number $c > 0$, such that for any*

$$a = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, there is a smooth family $(b(t))_{t \in [0,1]}$ of self-adjoint operators on \mathbb{C}^2 such that for $\lambda := |\lambda_1 - \lambda_2| + 1$ we have

$$\|b(t)\| \leq 2, \quad \|b'(t)\| \leq c\lambda, \quad b(0) = b(1) = 0, \quad q(1, 0)e_1 \in \text{span}(e_2).$$

where q is the evolution operator associated with $a + b(t)$, and e_i denotes the i^{th} standard unit vector.

Proof. Let $\phi : [0, 1] \rightarrow [0, \frac{\pi}{2}]$ be a smooth function (chosen independently of the λ_i) satisfying

$$|\phi'(t)| \leq 2, \quad \phi(0) = 0, \quad \phi(1) = \frac{\pi}{2}, \quad \phi'(0) = \phi'(1) = 0.$$

Consider the self-adjoint family

$$b(t) := \begin{pmatrix} 0 & i\phi'(t)\exp(i(\lambda_1 - \lambda_2)t) \\ -i\phi'(t)\exp(i(\lambda_2 - \lambda_1)t) & 0 \end{pmatrix}.$$

Then the evolution operator of $a + b(t)$ is given by

$$q(t, 0) := \begin{pmatrix} \exp(i\lambda_1 t)\cos(\phi(t)) & -\exp(i\lambda_1 t)\sin(\phi(t)) \\ \exp(i\lambda_2 t)\sin(\phi(t)) & \exp(i\lambda_2 t)\cos(\phi(t)) \end{pmatrix}.$$

Indeed, a straightforward calculation shows that $q(0, 0) = \text{Id}$ and

$$\frac{d}{dt}q(t, 0) = i(a + b(t))q(t, 0).$$

The required properties for b are easily checked, and the requirement $q(1, 0)e_1 \in \text{span}(e_2)$ follows since $q(1, 0)$ is off-diagonal. \square

PROPOSITION 5.14. *Let $\mathcal{H} := \bigoplus_{i=0}^{\infty} \mathbb{C}^2$ and let $(\lambda_i)_{i \geq 0}$ be an unbounded increasing sequence of positive real numbers. Consider the unbounded self-adjoint operator (with compact resolvents) given by*

$$A_0 := \bigoplus_{i=0}^{\infty} a_i, \quad a_i = \begin{pmatrix} -\lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix}.$$

There is a bounded family $B: [0, 1] \rightarrow B(\mathcal{H})$ such that $A(t) := A_0 + B(t)$ satisfies Assumption 5.1 and such that D_{APS} is not Fredholm.

Proofsketch. For $i \geq 0$, let b_i and q_i be chosen as b and q in Lemma 5.13 with $\lambda_1 = -\lambda_i$ and $\lambda_2 = \lambda_i$. Define

$$B := \bigoplus_{i=0}^{\infty} b_i, \quad Q := \bigoplus_{i=0}^{\infty} q_i.$$

Q is the evolution operator associated to the family $A(t) := A_0 + B(t)$. Let ι_i denote the inclusion of the i^{th} summand $\mathbb{C}^2 \hookrightarrow \mathcal{H}$. For all $i \in \mathbb{N}$, $\iota_i(e_1)$ is a negative eigenvector of $A(0) = A_0$, but

$$Q(1, 0)\iota_i(e_1) = \iota_i(q_i(1, 0)e_1) \in \text{span}(\iota_i(e_2))$$

is a positive eigenvector of $A(1) = A_0$ by construction. Thus

$$\text{Ker}(D_{\text{APS}}) \cong \text{Ker}(\widehat{P}_{<0}(1)_r) = \mathcal{H}_{<0}(0) \cap Q(0, 1)\mathcal{H}_{\geq 0}(1) = \overline{\{\text{span}\{\iota_i(e_1) \mid i \in \mathbb{N}\}\}}$$

is infinite-dimensional and hence D_{APS} is not Fredholm. \square

REFERENCES

- [1] M. F. ATIYAH, V. K. PATODI, AND I. M. SINGER, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
- [2] M. F. ATIYAH, V. K. PATODI, AND I. M. SINGER, *Spectral asymmetry and Riemannian geometry, III*, Math. Proc. Cambridge Philos. Soc. **79** (1976), 71–99.
- [3] J. AVRON, R. SEILER, AND B. SIMON, *The index of a pair of projections*, J. Funct. Anal. **120** (1994), no. 1, 220–237.
- [4] S. AZZALI AND C. WAHL, *Spectral flow, index and the signature operator*, J. Topol. Anal. **03** (2011), no. 01, 37–67.
- [5] C. BÄR AND A. STROHMAIER, *An index theorem for Lorentzian manifolds with compact spacelike Cauchy boundary*, Amer. J. Math. **141** (2019), no. 5, 1421–1455.
- [6] A. N. BERNAL AND M. SÁNCHEZ, *Smoothness of time functions and the metric splitting of globally hyperbolic space-times*, Commun. Math. Phys. **257** (2005), 43–50.
- [7] M. BRAVERMAN, *An index of strongly Callias operators on Lorentzian manifolds with non-compact boundary*, Math. Z. **294** (2020), 229–250.
- [8] J. BRÜNING AND M. LESCH, *On boundary value problems for Dirac type operators*, J. Funct. Anal. **185** (2001), no. 1, 1–62.
- [9] U. BUNKE AND T. HIRSCHMANN, *The index of the scattering operator on the positive spectral subspace*, Commun. Math. Phys. **148** (1992), 487–502.

- [10] A. L. CAREY, J. PHILLIPS, AND A. RENNIE, *A noncommutative Atiyah-Patodi-Singer index theorem in KK-theory*, *J. Reine Angew. Math.* **643** (2010), 59–109.
- [11] J. GELL-REDMAN, N. HABER, AND A. VASY, *The Feynman propagator on perturbations of Minkowski space*, *Commun. Math. Phys.* **342** (2016), 333–384.
- [12] E. HILLE AND R. PHILLIPS, *Functional analysis and semi-groups*, Colloquium publications, vol. 31, American Mathematical Society, 1996.
- [13] J. KAAD AND M. LESCH, *Spectral flow and the unbounded Kasparov product*, *Adv. Math.* **248** (2013), 495–530.
- [14] T. KATO, *Perturbation theory for linear operators*, corrected printing, second ed., Classics in Mathematics, Springer-Verlag, 1980.
- [15] M. LESCH, *The uniqueness of the spectral flow on spaces of unbounded self-adjoint Fredholm operators*, Spectral geometry of manifolds with boundary and decomposition of manifolds (B. Booss-Bavnbek, G. Grubb, and K. P. Wojciechowski, eds.), *Contemp. Math.*, vol. 366, Amer. Math. Soc., 2005, pp. 193–224.
- [16] A. PAZY, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer, New York, 1983.
- [17] J. PHILLIPS, *Self-adjoint Fredholm operators and spectral flow*, *Canad. Math. Bull.* **39** (1996), no. 4, 460–467.
- [18] J. ROBBIN AND D. SALAMON, *The spectral flow and the Maslov index*, *Bull. Lond. Math. Soc.* **27** (1995), no. 1, 1–33.
- [19] L. RONGE, *Index theory for globally hyperbolic spacetimes*, Master’s thesis, University of Bonn, 2019, arXiv:1910.10452.
- [20] M. STAROSTKA AND N. WATERSTRAAT, *On a comparison principle and the uniqueness of spectral flow*.
- [21] K. VAN DEN DUNGEN, *Families of spectral triples and foliations of space (time)*, *J. Math. Phys.* **59** (2018), no. 6, 063507.
- [22] K. VAN DEN DUNGEN, *The index of generalised Dirac-Schrödinger operators*, *J. Spectr. Theory* **9** (2019), 1459–1506.

(Received September 26, 2020)

Koen van den Dungen
Mathematisches Institut, Universität Bonn
Endenicher Allee 60, 53115 Bonn
e-mail: kdungen@uni-bonn.de

Lennart Ronge
Mathematisches Institut, Universität Bonn
Endenicher Allee 60, 53115 Bonn
e-mail: lennart.ronge@freenet.de