Operators and Matrices
Volume 15, Number 4 (2021), 1427–1444
doi:10.7153/oam-2021-15-89

SOME GENERAL QUADRATIC NUMERICAL RADIUS INEQUALITIES FOR THE OFF–DIAGONAL PARTS OF 2×2 BLOCK OPERATOR MATRICES

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(*Communicated by F. Kittaneh*)

Abstract. In this paper, the elementary properties for the quadratic numerical radius are introduced. Furthermore, some general quadratic numerical radius inequalities for the off-diagonal parts of 2×2 block operator matrices are studied. These inequalities are based on the generalized mixed Schwarz inequality, Young inequality and Jensen inequality.

1. Introduction

Let H_i, H_j be complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let $\mathbb{B}(H_i,H_j)$ denote the Banach space of all bounded linear operators from H_j into H_i, and abbreviate $\mathbb{B}(H_i,H_i)$ to $\mathbb{B}(H_i)$. For $T\in\mathbb{B}(H)$, we use T^*, $N(T)$ and $R(T)$ to denote the conjugate, the range space and the null space of T. The resolvent set $\rho(T)$ of T consists of the complex numbers λ such that $T-\lambda I$ is a bijection on H; the spectrum $\sigma(T)$ of T is the complement of $\rho(T)$ in $\mathbb{C}$. The point spectrum $\sigma_p(T)$, the residual spectrum $\sigma_r(T)$ and the numerical range $W(T)$ of T are the set

$$\sigma_p(T)=\{\lambda\in\mathbb{C}: T-\lambda I \text{ is not injective}\},$$

$$\sigma_r(T)=\{\lambda\in\mathbb{C}: T-\lambda I \text{ is injective}, \overline{R(T-\lambda I)}\neq H\},$$

$$W(T)=\{\langle Tx,x\rangle : x\in H, \|x\|=1\}.$$

The real numbers

$$\omega(T)=\sup_{\|x\|=1}|\langle Tx,x\rangle|,$$

$$\|T\|=\sup_{\|x\|=1}\|Tx\|$$

Mathematics subject classification (2020): 47A12.

Keywords and phrases: Bounded block operator matrix, quadratic numerical range, qadratic numerical radius, mixed Schwarz inequality, Young inequality, Jensen inequality.

This research is supported by the NNSF of China (Grant Nos. 11561048, 11761029), NSF of Inner Mongolia (Grant Nos. 2019MS01019, 2020ZD01).

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and

$$r(T) = \max_{\lambda \in \sigma(T)} |\lambda|$$

are called the numerical radius, the operator norm and the spectral radius, respectively.

By the well-known Toeplitz-Hausdorff theorem, the numerical range for bounded linear operators is always a convex subset of the complex plane, and it satisfies the so-called spectral inclusion property (see, e.g., [10]):

$$\sigma_p(T) \cup \sigma_r(T) \subset W(T), \sigma(T) \subset \overline{W(T)}.$$

However, the numerical range often gives a poor localization of the spectrum and it cannot capture finer structures such as the separation of the spectrum in two parts. In view of these shortcomings, the new concept of the quadratic numerical range was introduced in 1998 in [17] and further studied in [16, 18, 23–24].

Let H_1 and H_2 be Hilbert spaces. In the Hilbert space $H = H_1 \bigoplus H_2$, we consider the linear operator $\mathscr{A}$, given by the block operator matrix

$$\mathscr{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1}$$

with $A \in \mathbb{B}(H_1)$, $D \in \mathbb{B}(H_2)$, $B \in \mathbb{B}(H_2, H_1)$ and $C \in \mathbb{B}(H_1, H_2)$. Let $\sum$ denote the set

$$\sum := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : f \in H_1, g \in H_2, \|f\| = \|g\| = 1 \right\},$$

$\mathscr{A}_{f,g}$ the 2×2-matrix

$$\mathscr{A}_{f,g} = \begin{bmatrix} \langle Af, f\rangle & \langle Bg, f\rangle \\ \langle Cf, g\rangle & \langle Dg, g\rangle \end{bmatrix}, \quad f \in H_1, \ g \in H_2$$

and $\mathscr{I}_2$ the 2×2-identity matrix.

The set

$$W^2(\mathscr{A}) = \bigcup_{\|f\|=\|g\|=1} \sigma_p(\mathscr{A}_{f,g})$$

is called the quadratic numerical range of the operator $\mathscr{A}$ (with respect to the block operator representation (1)).

Obviously, $W^2(\mathscr{A})$ can also be written as

$$\begin{aligned} W^2(\mathscr{A}) &= \left\{ \lambda \in \mathbb{C} : det(\mathscr{A}_{f,g} - \lambda \mathscr{I}_2) = 0, \text{ for some } (f,g)^t \in \sum \right\} \\ &= \left\{ \lambda \in \mathbb{C} : det \begin{bmatrix} \langle Af, f\rangle - \lambda & \langle Bg, f\rangle \\ \langle Cf, g\rangle & \langle Dg, g\rangle - \lambda \end{bmatrix} = 0, \text{ for some } (f,g)^t \in \sum \right\} \\ &= \left\{ \lambda \in \mathbb{C} : \lambda = \frac{1}{2}\left[\langle Af, f\rangle + \langle Dg, g\rangle \pm \sqrt{|\Delta_{f,g}| e^{\frac{i}{2}\theta_{f,g}}} \right], \text{ for some } (f,g)^t \in \sum \right\}, \end{aligned}$$

where $\Delta_{f,g} = (\langle Af, f\rangle - \langle Dg, g\rangle)^2 + 4\langle Bg, f\rangle \langle Cf, g\rangle$. Here the superscript t denotes the transpose of a vector, $\theta_{f,g}$ denotes the argument of the complex number $\Delta_{f,g}$.

The quadratic numerical range is always contained in the numerical range (see, e.g., [23]):

$$W^2(\mathscr{A}) \subset W(\mathscr{A}).$$

However, unlike the numerical range, the quadratic numerical range is no longer convex;it consists of at most two components which need not be convex either. On the other hand, the quadratic numerical range shares other properties with the numerical range; it has the spectral inclusion property (see, e.g., [23]):

$$\sigma_p(\mathscr{A}) \cup \sigma_r(\mathscr{A}) \subset W^2(\mathscr{A}), \;\; \sigma(\mathscr{A}) \subset \overline{W^2(\mathscr{A})},$$

where $\mathscr{A}$ is a bounded block operator matrix.

We know that the numerical radius of a bounded linear operator is a powerful tool to characterize the spectral distribution, and more inequalities for the numerical radius can be found in [1–2, 4–5, 7–9, 11, 13–14, 19, 21–22]. For instance, Aghideh, Moslehian and Rooin in [1] proved that if $T = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, and f, g are non-negative non-decreasing continuous function on $[0,\infty)$ such that $f(t)g(t) = t \;\; (t \geqslant 0)$, then for all non-negative nondecreasing convex function h on $[0,\infty)$,

$$h(\omega(T)) \leqslant \frac{1}{4}\|h(f^2(|B|)) + h(g^2(|B|))\| + \frac{1}{4}\|h(f^2(|C|)) + h(g^2(|C|))\|.$$

The definition of the block numerical radius for bounded operator matrix was introduced in 2014 (see, e.g., [20]). In this paper, we only discuss the quadratic numerical radius. With respect to the block operator representation (1), then

$$\omega_2(\mathscr{A}) = \sup\{|\lambda| : \lambda \in W^2(\mathscr{A})\}$$

is called the quadratic numerical radius of the operator $\mathscr{A}$. Obviously, $\omega_2(\mathscr{A})$ can also be written as

$$\omega_2(\mathscr{A}) = \sup_{\|f\|=\|g\|=1} r(\mathscr{A}_{f,g}).$$

It is well-known that $\omega(\cdot)$ defines a norm on $\mathbb{B}(H)$, which is equivalent to the usual operator norm $\|\cdot\|$. Namely, for $\mathscr{A} \in \mathbb{B}(H)$, we have

$$\omega(\mathscr{A}) \leqslant \|\mathscr{A}\| \leqslant 2\omega(\mathscr{A}).$$

From the above inequalities (also see, e.g., [10]) and the relationship between the quadratic numerical range and the spectrum, we can obtain following inequalities:

$$r(\mathscr{A}) \leqslant \omega_2(\mathscr{A}) \leqslant \omega(\mathscr{A}) \leqslant \|\mathscr{A}\| \leqslant 2\omega(\mathscr{A}). \tag{2}$$

EXAMPLE 1. Let $\mathscr{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\|\mathscr{A}\| = 1$, $\omega_2(\mathscr{A}) = 0$, $\omega(\mathscr{A}) = \frac{1}{2}$, $\sigma(\mathscr{A}) = \{0\}$, hence the inequalities (2) hold and the quadratical numerical radius gives more precise information than the numerical radius in calculating the range of the spectrum.

In this paper, the elementary properties for the quadratic numerical radius are introduced. Furthermore, the main ideas of the general quadratic numerical radius inequalities for the off-diagonal parts of 2×2 block operator matrices are motivated by the references [1, 4–5, 11, 21]. These inequalities are based on the generalized mixed Schwarz inequality, Young inequality and Jensen inequality.

2. Preliminaries

In order to prove our results we need a sequence of lemmas. The first lemma is important and it has been proved by Kittaneh [15].

LEMMA 1. ([15]) *Let $M \in \mathbb{B}(H)$ and let h and k be nonnegative functions on $[0,\infty)$, which are continuous and satisfy the relation $h(s)k(s)=s$ for all $s\in[0,\infty)$. Then*

$$|\langle Mx,y\rangle| \leqslant \|h(|M|)x\|\|k(|M^*|)y\| \text{ for all } x,y\in H.$$

The next lemma follows from the spectral theorem for positive operators and Jensen's inequality (see, e.g., [15]).

LEMMA 2. ([15]) *Let A be a nonnegative bounded linear operator on a Hilbert space H, and let $x\in H$ be any unit vector. Then*

(i) $\langle Ax,x\rangle^r \leqslant \langle A^r x,x\rangle$ *for* $r\geqslant 1$;

(ii) $\langle A^r x,x\rangle \leqslant \langle Ax,x\rangle^r$ *for* $0<r\leqslant 1$.

The third lemma is a consequence of Young's inequality.

LEMMA 3. ([3]) (Young inequality)

(i) *(The classical Young inequality) If $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then for all positive real numbers a,b, we have*

$$ab \leqslant \frac{a^p}{p}+\frac{b^q}{q}. \tag{3}$$

(ii) *(Refinement of the scalar Young inequality) If $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then for all positive real numbers a,b, we have*

$$(a^{\frac{1}{p}}b^{\frac{1}{q}})^m + r_0^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2 \leqslant \left(\frac{a}{p}+\frac{b}{q}\right)^m,$$

where $r_0=\min\{\frac{1}{p},\frac{1}{q}\}$ and $m=1,2,\cdots$. In particular, if $p=q=2$, then

$$(a^{\frac{1}{2}}b^{\frac{1}{2}})^m + \left(\frac{1}{2}\right)^m (a^{\frac{m}{2}}-b^{\frac{m}{2}})^2 \leqslant 2^{-m}(a+b)^m. \tag{4}$$

The following lemma is a refinement of Schwarz's inequality (see, e.g., [6]).

LEMMA 4. ([6]) *Let* $(H, \langle\cdot,\cdot\rangle)$ *be real or complex Hilbert space over the real or complex field* $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\alpha \in \mathbb{K}$ *with* $|\alpha - 1| = 1$. *Then for any* $e \in H$ *with* $\|e\| = 1$ *and* $x, y \in H$, *we have*

$$|\langle x,y\rangle - \alpha\langle x,e\rangle\langle e,y\rangle| \leqslant \|x\|\|y\|.$$

The last lemma is a consequence of Jensen's inequality, concerning the convexity or the concavity of certain power functions. It is a special case of Schlömilch's inequality for weighted means of nonnegative real numbers (see, e.g., [12], p. 26).

LEMMA 5. ([12]) *For* $a, b \geqslant 0$, $0 < \alpha < 1$, *and* $r \neq 0$, *let* $M_r(a,b,\alpha) = (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}$ *and* $M_0(a,b,\alpha) = a^{\alpha} b^{1-\alpha}$. *Then*

$$M_r(a,b,\alpha) \leqslant M_s(a,b,\alpha), \quad r \leqslant s.$$

The numerical radius has some basic properties, for example, $\omega(\alpha T) = |\alpha|\omega(T)$ and $\omega(U^*TU) = \omega(T)$, where $\alpha \in \mathbb{C}$, $T \in \mathbb{B}(H)$, U is a unitary operator. Therefore we will consider the properties of the quadratic numerical radius.

PROPERTY 1. Let $\mathscr{A}$ be a bounded block operator matrix. Then
(i) $\omega_2(\mathscr{A}) = \omega_2(\mathscr{A}^*)$.
(ii) $\omega_2(\mathscr{A}) = \omega_2(\mathscr{U}^*\mathscr{A}\mathscr{U})$, where $\mathscr{U} = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$ or $\mathscr{U} = \begin{bmatrix} 0 & U_1 \\ U_2 & 0 \end{bmatrix}$, U_1 and U_2 are unitary operators.
(iii) $\omega_2(\alpha\mathscr{A}) = |\alpha|\omega_2(\mathscr{A})$, $\alpha \in \mathbb{C}$.
(iv) If $\mathscr{A}$ is selfadjoint, then $\omega_2(\mathscr{A}) = \|\mathscr{A}\|$.

Proof. Since $W^2(\mathscr{A}^*) = (W^2(\mathscr{A}))^*$ and $W^2(\mathscr{U}^*\mathscr{A}\mathscr{U}) = W^2(\mathscr{A})$, so the proof of (i) and (ii) are obvious. Part (iii) follows by applying the identity $W^2(\alpha\mathscr{A}) = \alpha W^2(\mathscr{A})$, where $\alpha \in \mathbb{C}$. (iv) If $\mathscr{A}$ is selfadjoint, then $\|\mathscr{A}^n\| = \|\mathscr{A}\|^n$. Moreover, recalling that $r(\mathscr{A}) = \lim_{n\to\infty} \|\mathscr{A}^n\|^{\frac{1}{n}}$, thus we have $r(\mathscr{A}) = \omega_2(\mathscr{A}) = \omega(\mathscr{A}) = \|\mathscr{A}\|$. □

PROPOSITION 1. Let $A, B, C, D \in \mathbb{B}(H)$. Then
(i) $\omega_2\left(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\right) = \omega_2\left(\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}\right) = \max\{\omega(A), \omega(D)\}$.
(ii) $\omega_2\left(\begin{bmatrix} 0 & B \\ e^{i\theta}C & 0 \end{bmatrix}\right) = \omega_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$, for all $\theta \in \mathbb{R}$.
(iii) $\omega_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \omega_2\left(\begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}\right)$.
(iv) $\omega_2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \max\{\omega(A+B), \omega(A-B)\}$.
(v) $\omega_2\left(\begin{bmatrix} A & B \\ -B & A \end{bmatrix}\right) = \max\{\omega(A+iB), \omega(A-iB)\}$.
(vi) $\omega_2\left(\begin{bmatrix} 0 & B \\ e^{i\theta}B & 0 \end{bmatrix}\right) = \omega(B)$.

Proof. (i) If $\mathscr{A} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ or $\mathscr{A} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$, then $W^2(\mathscr{A}) = W(A) \cup W(D)$, and thus $\omega_2(\mathscr{A}) = \max\{\omega(A), \omega(D)\}$.

Part (ii) follows by applying the property 1(ii) to the operator $\begin{bmatrix} 0 & B \\ e^{i\theta}C & 0 \end{bmatrix}$ and the unitary operator $\mathscr{U}_1 = \begin{bmatrix} I & 0 \\ 0 & e^{\frac{i\theta}{2}}I \end{bmatrix}$.

Part (iii) follows by applying the property 1(ii) to the operator $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ and the unitary operator $\mathscr{U}_2 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

(iv) Since

$$\begin{bmatrix} \frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \\ -\frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}I & -\frac{1}{\sqrt{2}}I \\ \frac{1}{\sqrt{2}}I & \frac{1}{\sqrt{2}}I \end{bmatrix} = \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix},$$

hence $\omega_2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \leqslant \omega\left(\begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix}\right) = \max\{\omega(A+B), \omega(A-B)\}$.

On the other hand,

$$\begin{aligned}
&\omega_2\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \\
&= \sup_{\|f\|=\|g\|=1} \frac{1}{2}|[\langle Af,f\rangle + \langle Ag,g\rangle \pm \sqrt{|(\langle Af,f\rangle - \langle Ag,g\rangle)^2 + 4\langle Bg,f\rangle\langle Bf,g\rangle|e^{\frac{i}{2}\theta_{f,g}}}]| \\
&\geqslant \frac{1}{2}\sup_{f=g,\|f\|=1} |2\langle Af,f\rangle \pm \sqrt{4|\langle Bf,f\rangle\langle Bf,f\rangle|}e^{\frac{i}{2}\theta_{f,f}}| \\
&= \sup_{\|f\|=1} |\langle Af,f\rangle \pm |\langle Bf,f\rangle| e^{\frac{i}{2}\theta_{f,f}}| \\
&= \sup_{\|f\|=1} |\langle Af,f\rangle \pm \langle Bf,f\rangle| \\
&= \max\{\omega(A+B), \omega(A-B)\}.
\end{aligned}$$

Thus we can obtain the result.

(v) By utilizing the similar proof of the (iv), we can obtain the result.

(vi) Taking $A = 0$ in (iv), then we can get the result. □

As we all know, if $T \in \mathbb{B}(H)$ with $\omega(T) = \|T\|$, then $r(T) = \omega(T)$ (see,e.g., [10]). So, we consider similar conclusion about the quadratic numerical radius.

PROPOSITION 2. If 2×2 bounded block operator matrix $\mathscr{A}$ satisfy $\|\mathscr{A}\| = \omega_2(\mathscr{A})$, then

$$r(\mathscr{A}) = \omega(\mathscr{A}) = \omega_2(\mathscr{A}) = \|\mathscr{A}\|.$$

Proof. Since $r(\mathscr{A}) \leqslant \omega_2(\mathscr{A}) \leqslant \omega(A) \leqslant \|\mathscr{A}\|$ and $\omega_2(\mathscr{A}) = \|\mathscr{A}\|$, so

$$\omega(\mathscr{A}) = \|\mathscr{A}\|,$$

which means that,

$$r(\mathscr{A}) = \omega_2(\mathscr{A}) = \omega(\mathscr{A}) = \|\mathscr{A}\|. \quad \square$$

For the numerical radius, inequality $\omega(\mathscr{A}) \leqslant \|\mathscr{A}\| \leqslant 2\omega(\mathscr{A})$ holds naturally, but for the quadratic numerical radius, it is not necessarily hold (see example 1).

PROPOSITION 3. If 2×2 bounded block operator matrix $\mathscr{A}$ satisfies $\omega_2(\mathscr{A} \pm \mathscr{A}^*) \leqslant \omega_2(\mathscr{A}) + \omega_2(\mathscr{A}^*)$, then $\omega_2(\mathscr{A}) \leqslant \|\mathscr{A}\| \leqslant 2\omega_2(\mathscr{A})$.

Proof. The inequality $\omega_2(\mathscr{A}) \leqslant \|\mathscr{A}\|$ is obvious and by the property 1(iv), we have

$$\begin{aligned}
\|\mathscr{A}\| &= \|Re\mathscr{A} + iIm\mathscr{A}\| \\
&\leqslant \|Re\mathscr{A}\| + \|Im\mathscr{A}\| \\
&= \omega_2(Re\mathscr{A}) + \omega_2(Im\mathscr{A}) \\
&= \frac{1}{2}[\omega_2(\mathscr{A} + \mathscr{A}^*) + \omega_2(\mathscr{A} - \mathscr{A}^*)] \\
&\leqslant 2\omega_2(\mathscr{A}). \quad \square
\end{aligned}$$

Also, the known result $\omega(T^r) \leqslant (\omega(T))^r$ for all $r = 1, 2, \cdots$, so we consider similar conclusion for the quadratical numerical radius.

PROPOSITION 4. Let $\mathscr{A} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(H \oplus H)$. Then $\omega_2(\mathscr{A}^{2r}) \leqslant [\omega_2(\mathscr{A})]^{2r}$, for $r = 1, 2, \cdots$.

Proof. Since

$$\mathscr{A}^{2r} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}^{2r} = \begin{bmatrix} (BC)^r & 0 \\ 0 & (CB)^r \end{bmatrix},$$

by the proposition 1(i) and the the power inequality of the numerical radius, we have

$$\begin{aligned}
[\omega_2(\mathscr{A})]^{2r} &= \sup_{\|f\|=\|g\|=1} |\langle Bg, f\rangle|^r |\langle Cf, g\rangle|^r \\
&\geqslant \max\{ \sup_{\substack{f=\frac{Bg}{\|Bg\|}, \\ Bg\neq 0,\ \|g\|=1}} |\langle Bg, f\rangle|^r |\langle Cf, g\rangle|^r, \sup_{\substack{g=\frac{Cf}{\|Cf\|}, \\ Cf\neq 0, \|f\|=1}} |\langle Bg, f\rangle|^r |\langle Cf, g\rangle|^r \} \\
&= \max\{[\omega(CB)]^r, [\omega(BC)]^r\} \\
&\geqslant \max\{\omega[(CB)^r], \omega[(BC)^r]\} \\
&= \omega_2 \begin{bmatrix} (BC)^r & 0 \\ 0 & (CB)^r \end{bmatrix} = \omega_2(\mathscr{A}^{2r}). \quad \square
\end{aligned}$$

3. Some general inequalities of the quadratic numerical radius for off-diagonal 2×2 block operator matrices

The aim of this section is to give some general inequalities of the quadratic numerical radius for the operator matrix $\mathscr{A} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Considering when $B = 0$ or $C = 0$, there is $\omega_2(\mathscr{A}) = 0$, then its lower bound must be 0. So we assume that $B \neq 0$ and $C \neq 0$.

THEOREM 1. *Let* $\mathscr{A} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(H \oplus H)$*, and let* h, k *be nonnegative functions on* $[0, \infty)$*, which are continuous and that satisfy the relation* $h(s)k(s) = s$ *for all* $s \in [0, \infty)$*, and* $r \geqslant 1$*. Then*

$$[\omega_2(\mathscr{A})]^r \leqslant \frac{1}{4}(\|h^{2r}(|B|) + k^{2r}(|C^*|)\| + \|k^{2r}(|B^*|) + h^{2r}(|C|)\|) - \frac{1}{4} \inf_{\|f\|=\|g\|=1} \xi(f,g)$$

and

$$[\omega_2(\mathscr{A})]^r \geqslant \frac{4 \max\{[\omega(BC)]^r, [\omega(CB)]^r\}}{\|h^{2r}(|B|) + k^{2r}(|C^*|)\| + \|k^{2r}(|B^*|) + h^{2r}(|C|)\| - \inf\limits_{\|f\|=\|g\|=1} \xi(f,g)},$$

where

$$\xi(f,g) = [\langle (h^{2r}(|B|) + k^{2r}(|C^*|))g, g\rangle^{\frac{1}{2}} - \langle (k^{2r}(|B^*|) + h^{2r}(|C|))f, f\rangle^{\frac{1}{2}}]^2.$$

Proof. Let f and g be any two unit vectors in H. Then using the elementary inequality

$$(ab + cd)^2 \leqslant (a^2 + c^2)(b^2 + d^2), a, b, c, d \in \mathbb{R}, \tag{5}$$

we have

$$\begin{aligned}
&|\langle Bg, f\rangle \langle Cf, g\rangle|^{\frac{r}{2}} = |\langle Bg, f\rangle|^{\frac{r}{2}} |\langle Cf, g\rangle|^{\frac{r}{2}} \\
&\leqslant \frac{1}{2}(|\langle Bg, f\rangle|^r + |\langle Cf, g\rangle|^r) \text{ (by the arithmetic-geomeiric inequality)} \\
&\leqslant \frac{1}{2}(\|h(|B|)g\|^r \|k(|B^*|)f\|^r + \|h(|C|)f\|^r \|k(|C^*|)g\|^r) \text{ (by lemma 1)} \\
&= \frac{1}{2}(\langle h^2(|B|)g, g\rangle^{\frac{r}{2}} \langle k^2(|B^*|)f, f\rangle^{\frac{r}{2}} + \langle h^2(|C|)f, f\rangle^{\frac{r}{2}} \langle k^2(|C^*|)g, g\rangle^{\frac{r}{2}}) \\
&\leqslant \frac{1}{2}(\langle h^{2r}(|B|)g, g\rangle^{\frac{1}{2}} \langle k^{2r}(|B^*|)f, f\rangle^{\frac{1}{2}} + \langle h^{2r}(|C|)f, f\rangle^{\frac{1}{2}} \langle k^{2r}(|C^*|)g, g\rangle^{\frac{1}{2}}) \text{ (by lemma 2)} \\
&\leqslant \frac{1}{2}(\langle h^{2r}(|B|)g, g\rangle + \langle k^{2r}(|C^*|)g, g\rangle)^{\frac{1}{2}} (\langle k^{2r}(|B^*|)f, f\rangle + \langle h^{2r}(|C|)f, f\rangle)^{\frac{1}{2}} \\
&\quad \text{(by the inequality (5))}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\langle (h^{2r}(|B|)+k^{2r}(|C^*|))g,g\rangle^{\frac{1}{2}}\langle (k^{2r}(|B^*|)+h^{2r}(|C|))f,f\rangle^{\frac{1}{2}} \\
&\leqslant \frac{1}{4}[\langle (h^{2r}(|B|)+k^{2r}(|C^*|))g,g\rangle + \langle (k^{2r}(|B^*|)+h^{2r}(|C|))f,f\rangle] \\
&\quad -\frac{1}{4}[\langle (h^{2r}(|B|)+k^{2r}(|C^*|))g,g\rangle^{\frac{1}{2}} - \langle (k^{2r}(|B^*|)+h^{2r}(|C|))f,f\rangle^{\frac{1}{2}}]^2 \\
&\qquad \text{(by the inequality (4))} \\
&\leqslant \frac{1}{4}(\|h^{2r}(|B|)+k^{2r}(|C^*|)\| + \|k^{2r}(|B^*|)+h^{2r}(|C|)\|) \\
&\quad -\frac{1}{4}[\langle (h^{2r}(|B|)+k^{2r}(|C^*|))g,g\rangle^{\frac{1}{2}} - \langle (k^{2r}(|B^*|)+h^{2r}(|C|))f,f\rangle^{\frac{1}{2}}]^2. \qquad (6)
\end{aligned}
$$

Thus taking the supremun over $f,g \in H$ with $\|f\| = \|g\| = 1$ in inequality (6), we have

$$
\begin{aligned}
[\omega_2(\mathscr{A})]^r &= \sup_{\|f\|=\|g\|=1} |\langle Bg,f\rangle\langle Cf,g\rangle|^{\frac{r}{2}} \\
&\leqslant \frac{1}{4}(\|h^{2r}(|B|)+k^{2r}(|C^*|)\| + \|k^{2r}(|B^*|)+h^{2r}(|C|)\|) - \frac{1}{4}\inf_{\|f\|=\|g\|=1}\xi(f,g),
\end{aligned}
$$

where

$$
\xi(f,g) = [\langle (h^{2r}(|B|)+k^{2r}(|C^*|))g,g\rangle^{\frac{1}{2}} - \langle (k^{2r}(|B^*|)+h^{2r}(|C|))f,f\rangle^{\frac{1}{2}}]^2.
$$

On the other hand, by the elementary inequality

$$
\sqrt{ab} \geqslant \frac{2}{\frac{1}{a}+\frac{1}{b}}, \quad a,b>0, \qquad (7)
$$

we have

$$
\begin{aligned}
&|\langle Bg,f\rangle\langle Cf,g\rangle|^{\frac{r}{2}} = |\langle Bg,f\rangle|^{\frac{r}{2}}|\langle Cf,g\rangle|^{\frac{r}{2}} \\
&\geqslant 2|\langle Bg,f\rangle|^r|\langle Cf,g\rangle|^r(|\langle Bg,f\rangle|^r+|\langle Cf,g\rangle|^r)^{-1} \quad \text{(by the inequality (7))} \\
&\geqslant 4|\langle Bg,f\rangle|^r|\langle Cf,g\rangle|^r[\|h^{2r}(|B|)+k^{2r}(|C^*|)\| + \|k^{2r}(|B^*|)+h^{2r}(|C|)\| \\
&-(\langle (h^{2r}(|B|)+k^{2r}(|C^*|))g,g\rangle^{\frac{1}{2}} - \langle (k^{2r}(|B^*|)+h^{2r}(|C|))f,f\rangle^{\frac{1}{2}})^2]^{-1} \qquad (8) \\
&\qquad \text{(by lemma 3 and lemma 1)} \\
&\geqslant 4|\langle Bg,f\rangle|^r|\langle Cf,g\rangle|^r[\|h^{2r}(|B|)+k^{2r}(|C^*|)\| + \|k^{2r}(|B^*|)+h^{2r}(|C|)\| \\
&\quad - \inf_{\|f\|=\|g\|=1}\xi(f,g)]^{-1}.
\end{aligned}
$$

Taking the supremun over $f,g \in H$ with $\|f\| = \|g\| = 1$ in inequality (8), we have

$$
\begin{aligned}
[\omega_2(\mathscr{A})]^r &= \sup_{\|f\|=\|g\|=1} |\langle Bg,f\rangle\langle Cf,g\rangle|^{\frac{r}{2}} \\
&\geqslant \frac{4\sup_{\|f\|=\|g\|=1}|\langle Bg,f\rangle|^r|\langle Cf,g\rangle|^r}{\|h^{2r}(|B|)+k^{2r}(|C^*|)\| + \|k^{2r}(|B^*|)+h^{2r}(|C|)\| - \inf_{\|f\|=\|g\|=1}\xi(f,g)}
\end{aligned}
$$

$$\geqslant \frac{4 \sup_{\|f\|=\|g\|=1, g=\frac{Cf}{\|Cf\|}} |\langle Bg, f\rangle|^r |\langle Cf, g\rangle|^r}{\|h^{2r}(|B|)+k^{2r}(|C^*|)\| + \|k^{2r}(|B^*|)+h^{2r}(|C|)\| - \inf_{\|f\|=\|g\|=1} \xi(f,g)}$$
$$= \frac{4[\omega(BC)]^r}{\|h^{2r}(|B|)+k^{2r}(|C^*|)\| + \|k^{2r}(|B^*|)+h^{2r}(|C|)\| - \inf_{\|f\|=\|g\|=1} \xi(f,g)}. \tag{9}$$

Similarly, taking $f = \frac{Bg}{\|Bg\|}$ in (9), we have

$$[\omega_2(\mathscr{A})]^r \geqslant \frac{4[\omega(CB)]^r}{\|h^{2r}(|B|)+k^{2r}(|C^*|)\| + \|k^{2r}(|B^*|)+h^{2r}(|C|)\| - \inf_{\|f\|=\|g\|=1} \xi(f,g)}.$$

This completes the proof. □

REMARK 1. If $f, g \in H$ with $\|f\| = \|g\| = 1$, by using the inequality

$$\begin{aligned} &|\langle Bg, f\rangle\langle Cf, g\rangle|^{\frac{r}{2}} = |\langle Bg, f\rangle|^{\frac{r}{2}} |\langle C^* g, f\rangle|^{\frac{r}{2}} \\ \leqslant &\frac{1}{2}(|\langle Bg, f\rangle|^r + |\langle C^* g, f\rangle|^r) \\ \leqslant &\frac{1}{2}(\|h(|B|)g\|^r \|k(|B^*|)f\|^r + \|h(|C^*|)g\|^r \|k(|C|)f\|^r) \end{aligned}$$

and the same argument in the proof of the theorem 1, we get the following inequalities:

$$[\omega_2(\mathscr{A})]^r \leqslant \frac{1}{4}(\|h^{2r}(|B|)+h^{2r}(|C^*|)\| + \|k^{2r}(|B^*|)+k^{2r}(|C|)\|) - \frac{1}{4} \inf_{\|f\|=\|g\|=1} \xi(f,g)$$

and

$$[\omega_2(\mathscr{A})]^r \geqslant \frac{4 \max\{[\omega(BC)]^r, [\omega(CB)]^r\}}{\|h^{2r}(|B|)+h^{2r}(|C^*|)\| + \|k^{2r}(|B^*|)+k^{2r}(|C|)\| - \inf_{\|f\|=\|g\|=1} \xi(f,g)},$$

where

$$\xi(f,g) = [\langle (h^{2r}(|B|)+h^{2r}(|C^*|))g, g\rangle^{\frac{1}{2}} - \langle (k^{2r}(|B^*|)+k^{2r}(|C|))f, f\rangle^{\frac{1}{2}}]^2.$$

In the next theorem, we show another general inequality for the quadratic numerical radius involving 2×2 off-diagonal block operator matrix.

THEOREM 2. *Let* $\mathscr{A} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(H \oplus H)$, *and let* h, k *be nonnegative functions on* $[0, \infty)$, *which are continuous and that satisfy the relation* $h(s)k(s) = s$ *for all* $s \in [0, \infty)$. *Then*

$$[\omega_2(\mathscr{A})]^{2r} \leqslant \|\frac{1}{p}h^{pr}(|B|) + \frac{1}{q}k^{qr}(|C^*|)\| \|\frac{1}{p}h^{pr}(|C|) + \frac{1}{q}k^{qr}(|B^*|)\|$$

and

$$[\omega_2(\mathscr{A})]^{2r} \geqslant \frac{2\max\{[\omega(BC)]^{2r},[\omega(CB)]^{2r}\}}{\|\frac{1}{p}h^{2rp}(|B|)+\frac{1}{q}k^{2rq}(|C^*|)\|+\|\frac{1}{q}k^{2rq}(|B^*|)+\frac{1}{p}h^{2rp}(|C|)\|},$$

for all $r \geqslant 1$, $p \geqslant q > 1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$ *and* $qr \geqslant 2$.

Proof. Let f and g be any two unit vectors in H. Then we have

$$\begin{aligned}
&|\langle Bg,f\rangle\langle Cf,g\rangle|^r = |\langle Bg,f\rangle|^r|\langle Cf,g\rangle|^r\\
\leqslant&\|h(|B|)g\|^r\|k(|B^*|)f\|^r\|h(|C|)f\|^r\|k(|C^*|)g\|^r \ \text{(by lemma 1)}\\
=&\langle h^2(|B|)g,g\rangle^{\frac{r}{2}}\langle k^2(|B^*|)f,f\rangle^{\frac{r}{2}}\langle h^2(|C|)f,f\rangle^{\frac{r}{2}}\langle k^2(|C^*|)g,g\rangle^{\frac{r}{2}}\\
=&\langle h^2(|B|)g,g\rangle^{\frac{r}{2}}\langle k^2(|C^*|)g,g\rangle^{\frac{r}{2}}\langle h^2(|C|)f,f\rangle^{\frac{r}{2}}\langle k^2(|B^*|)f,f\rangle^{\frac{r}{2}}\\
\leqslant&\left(\frac{1}{p}\langle h^2(|B|)g,g\rangle^{\frac{pr}{2}}+\frac{1}{q}\langle k^2(|C^*|)g,g\rangle^{\frac{qr}{2}}\right)\left(\frac{1}{p}\langle h^2(|C|)f,f\rangle^{\frac{pr}{2}}+\frac{1}{q}\langle k^2(|B^*|)f,f\rangle^{\frac{qr}{2}}\right)\\
&\quad\text{(by the inequality (3))}\\
\leqslant&\left(\frac{1}{p}\langle h^{pr}(|B|)g,g\rangle+\frac{1}{q}\langle k^{qr}(|C^*|)g,g\rangle\right)\left(\frac{1}{p}\langle h^{pr}(|C|)f,f\rangle+\frac{1}{q}\langle k^{qr}(|B^*|)f,f\rangle\right)\\
&\quad\text{(by lemma 2)}\\
=&\left\langle\left(\frac{1}{p}h^{pr}(|B|)+\frac{1}{q}k^{qr}(|C^*|)\right)g,g\right\rangle\left\langle\left(\frac{1}{p}h^{pr}(|C|)+\frac{1}{q}k^{qr}(|B^*|)\right)f,f\right\rangle\\
\leqslant&\left\|\frac{1}{p}h^{pr}(|B|)+\frac{1}{q}k^{qr}(|C^*|)\right\|\left\|\frac{1}{p}h^{pr}(|C|)+\frac{1}{q}k^{qr}(|B^*|)\right\|.
\end{aligned} \tag{10}$$

Taking the supremum over $f,g \in H$ with $\|f\|=\|g\|=1$ in inequality (10), we get the desired result.

On the other hand, we have

$$\begin{aligned}
&|\langle Bg,f\rangle\langle Cf,g\rangle|^r = |\langle Bg,f\rangle|^r|\langle Cf,g\rangle|^r\\
\geqslant& 2|\langle Bg,f\rangle|^{2r}|\langle Cf,g\rangle|^{2r}(|\langle Bg,f\rangle|^{2r}+|\langle Cf,g\rangle|^{2r})^{-1} \ \text{(by the inequality (7))}\\
\geqslant& \frac{2|\langle Bg,f\rangle|^{2r}|\langle Cf,g\rangle|^{2r}}{\|h(|B|)g\|^{2r}\|k(|B^*|)f\|^{2r}+\|h(|C|)f\|^{2r}\|k(|C^*|)g\|^{2r}} \ \text{(by lemma 1)}\\
=& \frac{2|\langle Bg,f\rangle|^{2r}|\langle Cf,g\rangle|^{2r}}{\langle h^2(|B|)g,g\rangle^r\langle k^2(|B^*|)f,f\rangle^r+\langle h^2(|C|)f,f\rangle^r\langle k^2(|C^*|)g,g\rangle^r}\\
\geqslant& \frac{2|\langle Bg,f\rangle|^{2r}|\langle Cf,g\rangle|^{2r}}{\frac{1}{p}\langle h^2(|B|)g,g\rangle^{rp}+\frac{1}{q}\langle k^2(|B^*|)f,f\rangle^{rq}+\frac{1}{p}\langle h^2(|C|)f,f\rangle^{rp}+\frac{1}{q}\langle k^2(|C^*|)g,g\rangle^{rq}}\\
&\quad\text{(by the inequality (3))}\\
\geqslant& \frac{2|\langle Bg,f\rangle|^{2r}|\langle Cf,g\rangle|^{2r}}{\frac{1}{p}\langle h^{2rp}(|B|)g,g\rangle+\frac{1}{q}\langle k^{2rq}(|B^*|)f,f\rangle+\frac{1}{p}\langle h^{2rp}(|C|)f,f\rangle+\frac{1}{q}\langle k^{2rq}(|C^*|)g,g\rangle}\\
&\quad\text{(by lemma 2)}
\end{aligned}$$

$$= \frac{2|\langle Bg,f\rangle|^{2r}|\langle Cf,g\rangle|^{2r}}{\langle(\frac{1}{p}h^{2rp}(|B|)+\frac{1}{q}k^{2rq}(|C^*|))g,g\rangle+\langle(\frac{1}{q}k^{2rq}(|B^*|)+\frac{1}{p}h^{2rp}(|C|))f,f\rangle}$$

$$\geqslant \frac{2|\langle Bg,f\rangle|^{2r}|\langle Cf,g\rangle|^{2r}}{\|\frac{1}{p}h^{2rp}(|B|)+\frac{1}{q}k^{2rq}(|C^*|)\|+\|\frac{1}{q}k^{2rq}(|B^*|)+\frac{1}{p}h^{2rp}(|C|)\|}. \tag{11}$$

Taking the supremum over $f,g \in H$ with $\|f\| = \|g\| = 1$ in (11) and the similar proof in (9), we get the desired result. □

REMARK 2. If $f,g \in H$ with $\|f\| = \|g\| = 1$, by using the inequality

$$\begin{aligned} &|\langle Bg,f\rangle\langle Cf,g\rangle|^r = |\langle Bg,f\rangle|^r|\langle C^*g,f\rangle|^r \\ \leqslant&\|h(|B|)g\|^r\|k(|B^*|)f\|^r\|h(|C^*|)g\|^r\|k(|C|)f\|^r \\ =&\langle h^2(|B|)g,g\rangle^{\frac{r}{2}}\langle k^2(|B^*|)f,f\rangle^{\frac{r}{2}}\langle h^2(|C^*|)g,g\rangle^{\frac{r}{2}}\langle k^2(|C|)f,f\rangle^{\frac{r}{2}}\end{aligned}$$

and the same argument in the proof of the theorem 2, we get the following inequalities:

$$[\omega_2(\mathscr{A})]^{2r} \leqslant \|\frac{1}{p}h^{pr}(|B|)+\frac{1}{q}h^{qr}(|C^*|)\|\|\frac{1}{p}k^{pr}(|B^*|)+\frac{1}{q}k^{qr}(|C|)\|$$

and

$$[\omega_2(\mathscr{A})]^{2r} \geqslant \frac{2\max\{[\omega(BC)]^{2r},[\omega(CB)]^{2r}\}}{\frac{1}{p}\|h^{2rp}(|B|)+h^{2rp}(|C^*|)\|+\frac{1}{q}\|k^{2rq}(|B^*|)+k^{2rq}(|C|)\|},$$

for all $r \geqslant 1$, $p \geqslant q > 1$ such that $\frac{1}{p}+\frac{1}{q} = 1$ and $qr \geqslant 2$.

REMARK 3. Let $h(s) = s^{\alpha}$, $k(s) = s^{1-\alpha}$, $\alpha \in [0,1]$ and $p = q = 2$ in the theorem 2. Then we get the following inequalities:

$$[\omega_2(\mathscr{A})]^{2r} \leqslant \frac{1}{4}\||B|^{2\alpha r}+|C^*|^{2(1-\alpha)r}\|\||B^*|^{2r(1-\alpha)}+|C|^{2r\alpha}\|,$$

and

$$[\omega_2(\mathscr{A})]^{2r} \geqslant \frac{4\max\{[\omega(BC)]^{2r},[\omega(CB)]^{2r}\}}{\||B|^{4r\alpha}+|C^*|^{4r(1-\alpha)}\|+\||B^*|^{4r(1-\alpha)}+|C|^{4r\alpha}\|}.$$

COROLLARY 1. *Let $B,C \in B(H)$. Then for all $0 \leqslant \alpha \leqslant 1$ and $r \geqslant 1$, we have*

$$[\omega(BC)]^r \leqslant \frac{1}{4}\||B|^{2\alpha r}+|C^*|^{2(1-\alpha)r}\|\||B^*|^{2r(1-\alpha)}+|C|^{2r\alpha}\|.$$

Proof. By utilizing the proposition 1 and proposition 4, we have

$$
\begin{aligned}
[\omega(BC)]^r &\leqslant \max\{[\omega(BC)]^r, [\omega(CB)]^r\}\\
&= \left(\omega_2 \begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix}\right)^r\\
&= \left(\omega_2 \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}^2\right)^r\\
&\leqslant \left(\omega_2 \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)^{2r}\\
&\leqslant \frac{1}{4}\||B|^{2\alpha r} + |C^*|^{2(1-\alpha)r}\|\||B^*|^{2r(1-\alpha)} + |C|^{2r\alpha}\|. \quad \square
\end{aligned}
$$

Following we show a different general inequality of the quadratic numerical radius involving 2×2 off-diagonal block operator matrix.

THEOREM 3. *Let* $\mathscr{A} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(H\oplus H)$, *and let* h,k *be nonnegative functions on* $[0,\infty)$, *which are continuous and that satisfy the relation* $h(s)k(s) = s$ *for all* $s \in [0,\infty)$. *Then*

$$[\omega_2(\mathscr{A})]^{2r} \leqslant \frac{1}{2}(\|\frac{1}{p}h^{pr}(|CB|) + \frac{1}{q}k^{qr}(|B^*C^*|)\| + \|B\|^r\|C\|^r)$$

and

$$[\omega_2(\mathscr{A})]^{2r} \geqslant \frac{2\max\{[\omega(BC)]^{2r}, [\omega(CB)]^{2r}\}}{\max\{\|B\|^{2r}, \|C\|^{2r}\} + \|\frac{1}{p}h^{pr}(|CB|) + \frac{1}{q}k^{qr}(|B^*C^*|)\|},$$

for all $r \geqslant 1$, $p \geqslant q > 1$ *with* $\frac{1}{p}+\frac{1}{q} = 1$ *and* $qr \geqslant 2$.

Proof. Let f and g be any two unit vectors in H. Then we have

$$
\begin{aligned}
|\langle Bg, f\rangle\langle Cf, g\rangle| &= |\langle Bg, f\rangle\langle f, C^*g\rangle|\\
&\leqslant \frac{1}{2}(|\langle CBg, g\rangle| + \|Bg\|\|C^*g\|) \text{ (by lemma 4)}\\
&\leqslant \left(\frac{1}{2}|\langle CBg, g\rangle|^r + \frac{1}{2}\|Bg\|^r\|C^*g\|^r\right)^{\frac{1}{r}} \text{ (by lemma 5),}
\end{aligned}
$$

hence we can obtain

$$
\begin{aligned}
&|\langle Bg, f\rangle\langle f, C^*g\rangle|^r \leqslant \frac{1}{2}(|\langle CBg, g\rangle|^r + \|Bg\|^r\|C^*g\|^r)\\
&\leqslant \frac{1}{2}(\|h(|CB|)g\|^r\|k(|B^*C^*|)g\|^r + \|B\|^r\|C\|^r) \text{ (by lemma 1)}\\
&= \frac{1}{2}(\langle h^2(|CB|)g, g\rangle^{\frac{r}{2}}\langle k^2(|B^*C^*|)g, g\rangle^{\frac{r}{2}} + \|B\|^r\|C\|^r)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{1}{2}(\frac{1}{p}\langle h^2(|CB|)g,g\rangle^{\frac{pr}{2}}+\frac{1}{q}\langle k^2(|B^*C^*|)g,g\rangle^{\frac{qr}{2}}+\|B\|^r\|C\|^r) \text{ (by the inequality (3))}\\
&\leqslant \frac{1}{2}(\frac{1}{p}\langle h^{pr}(|CB|)g,g\rangle+\frac{1}{q}\langle k^{qr}(|B^*C^*|)g,g\rangle+\|B\|^r\|C\|^r) \text{ (by lemma 2)}\\
&\leqslant \frac{1}{2}(\|\frac{1}{p}h^{pr}(|CB|)+\frac{1}{q}k^{qr}(|B^*C^*|)\|+\|B\|^r\|C\|^r).
\end{aligned}
\tag{12}
$$

Taking the supremum over $f,g\in H$ with $\|f\|=\|g\|=1$ in inequality (12), we get the desired result.

On the other hand, by the inequality (see, e.g., [7], p. 116)

$$
|\langle y,u\rangle|^2+|\langle y,v\rangle|^2\leqslant \|y\|^2[\max\{\|u\|^2,\|v\|^2\}+|\langle u,v\rangle|], \text{ for any } y,u,v\in H, \tag{13}
$$

and for any $\|f\|=\|g\|=1$, we have

$$
\begin{aligned}
&|\langle Bg,f\rangle\langle Cf,g\rangle| = |\langle f,Bg\rangle||\langle f,C^*g\rangle|\\
\geqslant\ &2|\langle f,Bg\rangle|^2|\langle f,C^*g\rangle|^2(|\langle f,Bg\rangle|^2+|\langle f,C^*g\rangle|^2)^{-1} \text{ (by the inequality (7))}\\
\geqslant\ &2|\langle f,Bg\rangle|^2|\langle f,C^*g\rangle|^2[\|f\|^2(\max\{\|Bg\|^2,\|C^*g\|^2\}+|\langle Bg,C^*g\rangle|)]^{-1}\\
&\quad \text{(by the inequality (13))}\\
=\ &\frac{2|\langle f,Bg\rangle|^2|\langle f,C^*g\rangle|^2}{\max\{\|Bg\|^2,\|C^*g\|^2\}+|\langle CBg,g\rangle|}\\
\geqslant\ &\frac{2|\langle f,Bg\rangle|^2|\langle f,C^*g\rangle|^2}{\max\{\|B\|^2,\|C\|^2\}+|\langle CBg,g\rangle|}\\
\geqslant\ &\frac{2|\langle f,Bg\rangle|^2|\langle f,C^*g\rangle|^2}{\max\{\|B\|^2,\|C\|^2\}+\|h(|CB|)g\|\|k(|B^*C^*|)g\|} \text{ (by lemma 1)}\\
=\ &\frac{|\langle f,Bg\rangle|^2|\langle f,C^*g\rangle|^2}{\frac{1}{2}\max\{\|B\|^2,\|C\|^2\}+\frac{1}{2}\langle h^2(|CB|)g,g\rangle^{\frac{1}{2}}\langle k^2(|B^*C^*|)g,g\rangle^{\frac{1}{2}}}\\
\geqslant\ &\frac{|\langle f,Bg\rangle|^2|\langle f,C^*g\rangle|^2}{(\frac{1}{2}\max\{\|B\|^{2r},\|C\|^{2r}\}+\frac{1}{2}\langle h^2(|CB|)g,g\rangle^{\frac{r}{2}}\langle k^2(|B^*C^*|)g,g\rangle^{\frac{r}{2}})^{\frac{1}{r}}} \text{ (by lemma 5)}\\
\geqslant\ &\frac{|\langle f,Bg\rangle|^2|\langle f,C^*g\rangle|^2}{(\frac{1}{2}\max\{\|B\|^{2r},\|C\|^{2r}\}+\frac{1}{2p}\langle h^2(|CB|)g,g\rangle^{\frac{pr}{2}}+\frac{1}{2q}\langle k^2(|B^*C^*|)g,g\rangle^{\frac{qr}{2}})^{\frac{1}{r}}}\\
&\quad \text{(by the inequality (3))}\\
\geqslant\ &\frac{|\langle f,Bg\rangle|^2|\langle f,C^*g\rangle|^2}{(\frac{1}{2}\max\{\|B\|^{2r},\|C\|^{2r}\}+\frac{1}{2p}\langle h^{pr}(|CB|)g,g\rangle+\frac{1}{2q}\langle k^{qr}(|B^*C^*|)g,g\rangle)^{\frac{1}{r}}} \text{ (by lemma 2)}\\
=\ &\frac{|\langle f,Bg\rangle|^2|\langle f,C^*g\rangle|^2}{(\frac{1}{2}\max\{\|B\|^{2r},\|C\|^{2r}\}+\langle(\frac{1}{2p}h^{pr}(|CB|)+\frac{1}{2q}k^{qr}(|B^*C^*|))g,g\rangle)^{\frac{1}{r}}}\\
\geqslant\ &\frac{|\langle f,Bg\rangle|^2|\langle f,C^*g\rangle|^2}{(\frac{1}{2}\max\{\|B\|^{2r},\|C\|^{2r}\}+\|\frac{1}{2p}h^{pr}(|CB|)+\frac{1}{2q}k^{qr}(|B^*C^*|)\|)^{\frac{1}{r}}},
\end{aligned}
$$

hence

$$|\langle Bg,f\rangle\langle f,C^{*}g\rangle|^{r}\geqslant\frac{2|\langle f,Bg\rangle|^{2r}|\langle f,C^{*}g\rangle|^{2r}}{\max\{\|B\|^{2r},\|C\|^{2r}\}+\|\frac{1}{p}h^{pr}(|CB|)+\frac{1}{q}k^{qr}(|B^{*}C^{*}|)\|}. \tag{14}$$

Taking the supremum over $f,g\in H$ with $\|f\|=\|g\|=1$ in inequality (14) and the similar proof in (9), we get the desired result. $\square$

REMARK 4. If $f,g\in H$ with $\|f\|=\|g\|=1$, by using the inequality

$$\begin{aligned}|\langle Bg,f\rangle\langle Cf,g\rangle| &= |\langle B^{*}f,g\rangle\langle g,Cf\rangle|\\ &\leqslant\frac{1}{2}(|\langle C^{*}B^{*}f,f\rangle|+\|B^{*}f\|\|Cf\|)\\ &\leqslant(\frac{1}{2}|\langle C^{*}B^{*}f,f\rangle|^{r}+\frac{1}{2}\|B^{*}f\|^{r}\|Cf\|^{r})^{\frac{1}{r}},\end{aligned}$$

and the same argument in the proof of the theorem 3, we get the following inequality:

$$[\omega_{2}(\mathscr{A})]^{2r}\leqslant\frac{1}{2}\Big(\Big\|\frac{1}{p}h^{pr}(|C^{*}B^{*}|)+\frac{1}{q}k^{qr}(|BC|)\Big\|+\|B\|^{r}\|C\|^{r}\Big).$$

By using the inequality

$$|\langle Bg,f\rangle\langle Cf,g\rangle|=|\langle g,B^{*}f\rangle||\langle g,Cf\rangle|$$

and the same argument in the proof of the theorem 3, we get the following inequality:

$$[\omega_{2}(\mathscr{A})]^{2r}\geqslant\frac{2\max\{[\omega(BC)]^{2r},[\omega(CB)]^{2r}\}}{\max\{\|B\|^{2r},\|C\|^{2r}\}+\|\frac{1}{p}h^{pr}(|BC|)+\frac{1}{q}k^{qr}(|C^{*}B^{*}|)\|}.$$

REMARK 5. Let $B=C$ in the theorem 3 and utilizing the proposition 1. Then we get the following inequality:

$$[\omega(B)]^{2r}=[\omega_{2}(\mathscr{A})]^{2r}\leqslant\frac{1}{2}\Big(\Big\|\frac{1}{p}h^{pr}(|B^{2}|)+\frac{1}{q}k^{qr}(|(B^{*})^{2}|)\Big\|+\|B\|^{2r}\Big),$$

for all $r\geqslant 1, p\geqslant q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $qr\geqslant 2$. This inequality is given in the proposition 2.5 in [21].

Using the similar proof of the corollary 2, we obtain the following corollary.

COROLLARY 2. *Let $B,C\in B(H)$. Then for all $0\leqslant\alpha\leqslant 1$ and $r\geqslant 1$, we have*

$$[\omega(BC)]^{r}\leqslant\frac{1}{4}\||CB|^{2\alpha r}+|B^{*}C^{*}|^{2(1-\alpha)r}\|+\frac{1}{2}\|B\|^{r}\|C\|^{r}.$$

Finally, we will give the upper bound of the $[\omega_{2}(\mathscr{A})]^{4r}$.

THEOREM 4. *Let* $\mathscr{A} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathbb{B}(H \oplus H)$. *Then for all* $r \geqslant 1$, *we have*

$$[\omega_2(\mathscr{A})]^{4r} \leqslant \frac{3}{8}\||B|^{4r} + |C^*|^{4r}\| + \frac{1}{8}\||B|^{2r} + |C^*|^{2r}\|[\omega(CB)]^r.$$

Proof. Let f and g be any two unit vectors in H. Then using the inequality (see, e.g., [9])

$$|\langle x,e\rangle\langle e,y\rangle|^2 \leqslant \frac{3}{4}\|x\|^2\|y\|^2 + \frac{1}{4}\|x\|\|y\||\langle x,y\rangle|, \text{ for any } x,y,e \in H \text{ and } \|e\| = 1, \quad (15)$$

we have

$$\begin{aligned} |\langle Bg,f\rangle\langle Cf,g\rangle|^2 &= |\langle Bg,f\rangle\langle f,C^*g\rangle|^2 \\ &\leqslant \frac{3}{4}\|Bg\|^2\|C^*g\|^2 + \frac{1}{4}|\langle CBg,g\rangle|\|Bg\|\|C^*g\| \text{ (by the inequality (15))} \\ &\leqslant (\frac{3}{4}\|Bg\|^{2r}\|C^*g\|^{2r} + \frac{1}{4}|\langle CBg,g\rangle|^r\|Bg\|^r\|C^*g\|^r)^{\frac{1}{r}}, \text{ (by lemma 5)} \end{aligned}$$

hence

$$\begin{aligned} &|\langle Bg,f\rangle\langle f,C^*g\rangle|^{2r} \\ \leqslant&\frac{3}{4}\|Bg\|^{2r}\|C^*g\|^{2r} + \frac{1}{4}\|Bg\|^r\|C^*g\|^r|\langle CBg,g\rangle|^r \\ =&\frac{3}{4}\langle B^*Bg,g\rangle^r\langle CC^*g,g\rangle^r + \frac{1}{4}\langle B^*Bg,g\rangle^{\frac{r}{2}}\langle CC^*g,g\rangle^{\frac{r}{2}}|\langle CBg,g\rangle|^r \\ =&\frac{3}{4}\langle |B|^2g,g\rangle^r\langle |C^*|^2g,g\rangle^r + \frac{1}{4}\langle |B|^2g,g\rangle^{\frac{r}{2}}\langle |C^*|^2g,g\rangle^{\frac{r}{2}}|\langle CBg,g\rangle|^r \\ \leqslant&\frac{3}{8}(\langle |B|^2g,g\rangle^{2r} + \langle |C^*|^2g,g\rangle^{2r}) + \frac{1}{8}(\langle |B|^2g,g\rangle^r + \langle |C^*|^2g,g\rangle^r)|\langle CBg,g\rangle|^r \\ \leqslant&\frac{3}{8}(\langle |B|^{4r}g,g\rangle + \langle |C^*|^{4r}g,g\rangle) + \frac{1}{8}(\langle |B|^{2r}g,g\rangle + \langle |C^*|^{2r}g,g\rangle)|\langle CBg,g\rangle|^r \text{ (by lemma 2)} \\ \leqslant&\frac{3}{8}\||B|^{4r} + |C^*|^{4r}\| + \frac{1}{8}\||B|^{2r} + |C^*|^{2r}\|[\omega(CB)]^r. \end{aligned} \quad (16)$$

Taking the supremum over $f,g \in H$ with $\|f\| = \|g\| = 1$ in inequality (16), we get the desired result. □

REMARK 6. If $f,g \in H$ with $\|f\| = \|g\| = 1$, then by using the inequality

$$\begin{aligned} |\langle Bg,f\rangle\langle Cf,g\rangle|^2 &= |\langle B^*f,g\rangle\langle g,Cf\rangle|^2 \\ &\leqslant \frac{3}{4}\|B^*f\|^2\|Cf\|^2 + \frac{1}{4}|\langle BCf,f\rangle|\|B^*f\|\|Cf\| \end{aligned}$$

and the same argument in the proof of the theorem 4, we get the following inequality:

$$[\omega_2(\mathscr{A})]^{4r} \leqslant \frac{3}{8}\||B^*|^{4r} + |C|^{4r}\| + \frac{1}{8}\||B^*|^{2r} + |C|^{2r}\|[\omega(BC)]^r.$$

REMARK 7. Let $B=C$, $r=1$ in the theorem 4 and utilizing the proposition 1. Then we get the following inequality

$$[\omega(B)]^4=[\omega_2(\mathscr{A})]^4\leqslant\frac{3}{8}\||B|^4+|B^*|^4\|+\frac{1}{8}\||B|^2+|B^*|^2\|\omega(B^2).$$

This inequality is given in the theorem 2.1 in [9].

Acknowledgements. The authors are grateful to the referees for her/his useful suggestions.

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(Received October 7, 2020)

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