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DAVIS–WIELANDT–BEREZIN RADIUS INEQUALITIES VIA DRAGOMIR INEQUALITIES

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Abstract. We consider operator A on the reproducing Kernel Hilbert space $\mathscr{H}=\mathscr{H}(\Omega)$ over some set Ω with the reproducing kernel $\mathscr{K}_\lambda(z)=\mathscr{K}(z,\lambda)$ and define Davis-Wielandt-Berezin radius $\eta(A)$ by the formula

$$\eta(A):=\sup\left\{\sqrt{\left|\widetilde{A}(\lambda)\right|^2+\|A\mathscr{K}_\lambda\|^4}:\lambda\in\Omega\right\},$$

where $\widetilde{A}$ is the Berezin symbol of A defined by $\widetilde{A}(\lambda):=\left\langle A\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle$, $\lambda\in\Omega$, where $\widehat{\mathscr{K}_\lambda}=\frac{\mathscr{K}_\lambda}{\|\mathscr{K}_\lambda\|_{\mathscr{H}}}$ is the normalized reproducing kernel of $\mathscr{H}$. We prove several inequalities for this new quantity $\eta(A)$ involving known Dragomir inequalities. Some other Berezin number inequalities are also proved.

1. Introduction

Let $\mathscr{H}=\mathscr{H}(\Omega)$ be a reproducing kernel Hilbert space on some set Ω with the reproducing kernel $\mathscr{K}_\lambda\in\mathscr{H}$, i.e., $f(\lambda)=\langle f,\mathscr{K}_\lambda\rangle$ for all $\lambda\in\Omega$. Let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$ with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. For an operator $A\in\mathscr{B}(\mathscr{H})$, its Berezin symbol (Berezin [3, 4]) $\widetilde{A}$ is defined by $\widetilde{A}(\lambda):=\left\langle A\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle$, $\lambda\in\Omega$, where $\widehat{\mathscr{K}_\lambda}=\frac{\mathscr{K}_\lambda}{\|\mathscr{K}_\lambda\|_{\mathscr{H}}}$ is the normalized reproducing kernel of $\mathscr{H}$. The Berezin number of operator A is defined by (see Karaev [19, 20])

$$\operatorname{ber}(A):=\sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|.$$

The Berezin set and the Berezin norm of operator are defined, respectively, by

$$\operatorname{Ber}(A):=\operatorname{Range}\left(\widetilde{A}\right)\text{ and }\|A\|_{\operatorname{Ber}}:=\sup_{\lambda\in\Omega}\left\|A\widehat{\mathscr{K}_\lambda}\right\|.$$

It is clear that $\operatorname{Ber}(A)\subseteq W(A)$ (numerical range),

$$\operatorname{ber}(A)\leqslant w(A)\text{ (numerical radius) and }\operatorname{ber}(A)\leqslant\|A\|_{\operatorname{Ber}}.$$

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Motivated by theoretical study and applications, there have been many generalizations of the numerical radius; see [5, 17, 23, 25, 33, 34] One of these generalizations is the Davis-Wielandt radius of $A \in \mathscr{B}(\mathscr{H})$ defined by

$$dw(A) := \sup\left\{\sqrt{|\langle Ax, x\rangle|^2 + \|Ax\|^4} : x \in \mathscr{H} \text{ and } \|x\| = 1\right\};$$

see [6, 26, 34].

DEFINITION 1. For any $A \in \mathscr{B}(\mathscr{H})$, we define its Davis-Wielandt-Berezin radius by the formula

$$\eta(A) := \sup_{\lambda \in \Omega}\sqrt{\left|\widehat{A}(\lambda)\right|^2 + \left\|A\widehat{\mathscr{K}_\lambda}\right\|^4}.$$

It is obvious that $\eta(A) \leqslant dw(A)$. For $A, B \in \mathscr{B}(\mathscr{H})$ one has

(i) $\eta(A) \geqslant 0$ and $\eta(A) = 0$ if and only if $A = 0$;

(ii) $\eta(\alpha A)\begin{cases} \geqslant |\alpha|\eta(A) \text{ if } |\alpha| > 1 \\ = |\alpha|\eta(A) \text{ if } |\alpha| = 1 \text{ for all } \alpha \in \mathbb{C} \\ \leqslant |\alpha|\eta(A) \text{ if } |\alpha| < 1. \end{cases}$

(iii) $\eta(A+B) \leqslant \sqrt{2\left(\eta(A) + \eta(B) + 4(\eta(A) + \eta(B))^2\right)}$;

and therefore $\eta(\cdot)$ cannot be a norm on $\mathscr{B}(\mathscr{H})$. The following property of $\eta(\cdot)$ is immediate:

$$\max\left\{\operatorname{ber}(A), \|A\|_{\operatorname{Ber}}^2\right\} \leqslant \eta(A) \leqslant \sqrt{\operatorname{ber}^2(A) + \|A\|_{\operatorname{Ber}}^4}(A \in \mathscr{B}(\mathscr{H})).$$

The purpose of this paper is to establish some upper bounds for the Davis-Wielandt-Berezin radius of reproducing kernel Hilbert space operators. For this aim, we use some known Dragomir inequalities for vectors in inner product spaces (see Dragomir [7, 9, 10]). We also prove some new Berezin number inequalities for the quadratic weighted operator mean of (A_1, A_2) where A_1 and A_2 have similar positive parts.

2. Prerequisites

In the present section, we collect some auxiliary lemmas including Kittaneh and Manasrah [21] inequality and Dragomir [9, 10] inequalities.

LEMMA 1. ([18]) *Let $a, b \geqslant 0$, $0 \leqslant m, n \leqslant 1$ and $p, q > 1$ such that $m+n = 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

(i) $a^m b^n \leqslant ma + nb \leqslant (ma^r + nb^r)^{\frac{1}{r}}$ *for* $r \geqslant 1$*;*

(ii) $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q} \leqslant \left(\frac{a^{pr}}{p} + \frac{b^{qr}}{q}\right)^{\frac{1}{r}}$ *for* $r \geqslant 1$.

LEMMA 2. ([21]) *Let $a,b \geqslant 0$, and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$ab+\min\left\{\frac{1}{p},\frac{1}{q}\right\}\left(a^{\frac{p}{2}}-b^{\frac{q}{2}}\right)^2 \leqslant \frac{a^p}{p}+\frac{b^q}{q}.$$

Let $\mathscr{B}^{-1}(\mathscr{H})$ be the class of all bounded linear invertible operators on $\mathscr{H}$. In 2016, Dragomir [9] first introduced the concept of quadratic weighted operator geometric mean of operators. The same notion is recalled next. For $A_1 \in \mathscr{B}^{-1}(\mathscr{H})$ and $A_2 \in \mathscr{B}(\mathscr{H})$, the quadratic weighted operator geometric mean of (A_1,A_2) is defined by (see [24])

$$A_1 Ⓢ_\nu A_2 = \left|\left|A_2A_1^{-1}\right|^\nu A_1\right|^2 \text{ for } \nu \geqslant 0. \tag{1}$$

Using this mean, Dragomir obtained some fundamental inequalities for some class of operators. In 2018, he [10] again continued the same study and presented some Hölder type inequalities for the quadratic weighted operator geometric mean for the operators on the Hilbert space $\mathscr{H}$.

By using the definition of modulus, equation (1) can also be written as following:

$$A_1 Ⓢ_\nu A_2 = A_1^*\left|A_2A_1^{-1}\right|^{2\nu}A_1 = A_1^*\left(\left(A_1^*\right)^{-1}A_2^*A_2A_1^{-1}\right)^\nu A_1. \tag{2}$$

For $A_1 \in \mathscr{B}^{-1}(\mathscr{H})$, $A_2 \in \mathscr{B}(\mathscr{H})$ and $\nu=\frac{1}{2}$, equation (2) reduces to the following form (see [24]):

$$A_1 Ⓢ A_2 = \left|\left|A_2A_1^{-1}\right|^{\frac{1}{2}}A_1\right|^2 = A_1^*\left|A_2A_1^{-1}\right|A_1 = A_1^*\left(\left(A_1^*\right)^{-1}A_2^*A_2A_1^{-1}\right)^{\frac{1}{2}}A_1. \tag{3}$$

LEMMA 3. ([10]) *Let $A_1 \in \mathscr{B}^{-1}(\mathscr{H})$ and $A_2 \in \mathscr{B}(\mathscr{H})$. Then for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$*

$$\langle A_1 Ⓢ_{1/p} A_2x,x\rangle \leqslant \left\langle |A_2|^2x,x\right\rangle^{\frac{1}{p}}\left\langle |A_1|^2x,x\right\rangle^{\frac{1}{q}}$$

for any $x\in\mathscr{H}$. In particular,

$$\langle A_1 Ⓢ A_2x,x\rangle \leqslant \left\langle |A_2|^2x,x\right\rangle^{\frac{1}{2}}\left\langle |A_1|^2x,x\right\rangle^{\frac{1}{2}}$$

for any $x\in\mathscr{H}$.

Dragomir proved the following inequalities.

LEMMA 4. ([34]) *For any $a,b,c\in\mathscr{H}$ with $\|c\|=1$*

$$|\langle a,c\rangle\langle c,b\rangle| \leqslant \frac{1}{2}\left(|\langle a,b\rangle|+\|a\|\,\|b\|\right).$$

LEMMA 5. ([34]) *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$\frac{1}{2}\|Ax\| \leqslant \sqrt{\frac{w^2(A)}{2}+\frac{w(A)}{2}\sqrt{w^2(A)-|\langle Ax,x\rangle|^2}}$$

for any $x \in H$ with $\|x\| \leqslant 1$.

Recall that (see [28, 27]) the Crawford-Berezin number of $A \in \mathscr{B}(\mathscr{H})$ is defined by

$$\widetilde{c}(A) := \inf_{\lambda \in \Omega}\left|\widetilde{A}(\lambda)\right|.$$

The following lemma, due to Dragomir [8], is an extension of the Cauchy-Schwarz inequality for vectors in inner product spaces.

LEMMA 6. ([34]) *For any $a,b,c \in \mathscr{H}$*

$$|\langle a,b\rangle|^2+|\langle a,c\rangle|^2 \leqslant \|a\|^2\left(|\langle b,b\rangle|^2+2|\langle b,c\rangle|^2+|\langle c,c\rangle|^2\right)^{\frac{1}{2}}.$$

Another extensions of the Cauchy-Schwarz inequality for vectors in inner product space are the following results of Dragomir [8].

LEMMA 7. ([8]) *For any $a,b,c \in \mathscr{H}$, we have*

$$|\langle a,b\rangle|^2+|\langle a,c\rangle|^2 \leqslant \|a\|^2\left(\max\left\{\|b\|^2,\|c\|^2\right\}+\sqrt{2}|\langle b,c\rangle|\right).$$

3. The results

3.1. Dawis-Wielandt-Berezin radius inequalities

In this section, we apply lemmas in previous section, including Dragomir's inequalities, to prove new inequalities for Davis-Wielandt-Berezin radius of operators on $\mathscr{H} = \mathscr{H}(\Omega)$. The similar results for Davis-Wielandt radius of operators are contained in [34].

Let $\operatorname{ran}(A)$ and $\ker(A)$ denote the range space and the null space of an operator $A \in \mathscr{B}(\mathscr{H})$, respectively. An operator $A \in \mathscr{B}(\mathscr{H})$ is called a partial isometry if $A|_{(\ker(A))^{\perp}}$ is an isometry. If $A \in \mathscr{B}(\mathscr{H})$, then there is a unique polar decomposition of A given by $A = V_A|A|$, where V_A is a partial isometry, $|A| := (A^*A)^{\frac{1}{2}}$, $\ker(V_A) = \ker(|A|) = \ker(A)$ and $\ker(V_A^*) = \ker(A^*)$, where A^* represents the adjoint of the operator A. Ko [22] first discussed about the operators having similar positive parts. The operators A and B in $\mathscr{B}(\mathscr{H})$ is said to have similar positive parts if there exists an unitary operator $U \in \mathscr{B}(\mathscr{H})$ such that

$$U|B| = |A|U.$$

THEOREM 1. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$\eta^2(A) \leqslant \mathrm{ber}^2\left(|A|^2 - A\right) + 2\|A\|_{\mathrm{Ber}}^2 \,\mathrm{ber}(A).$$

Proof. For any $\lambda \in \Omega$, we have

$$\begin{aligned}
&\left|\left\langle A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 + \left\|A\widehat{\mathscr{K}_\lambda}\right\|^4 \\
&= \left|\left\langle A\widehat{\mathscr{K}_\lambda}, A\widehat{\mathscr{K}_\lambda}\right\rangle - \left\langle A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 + 2\mathrm{Re}\left(\left\langle A\widehat{\mathscr{K}_\lambda}, A\widehat{\mathscr{K}_\lambda}\right\rangle \left\langle A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right) \\
&= \left|\left\langle \left(|A|^2 - A\right)\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 + 2\left\|A\widehat{\mathscr{K}_\lambda}\right\|^2 \mathrm{Re}\left\langle A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle \\
&\leqslant \mathrm{ber}^2\left(|A|^2 - A\right) + 2\|A\|_{\mathrm{Ber}}^2 \,\mathrm{ber}(A),
\end{aligned}$$

hence, taking supremum over $\lambda \in \Omega$ gives

$$\eta^2(A) \leqslant \mathrm{ber}^2\left(|A|^2 - A\right) + 2\|A\|_{\mathrm{Ber}}^2 \,\mathrm{ber}(A),$$

as desired. □

The next result is the following.

THEOREM 2. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$\eta^2(A) \leqslant \frac{1}{2}\mathrm{ber}\left(|A|^2 + 2|A|^4 + |A^*|^2\right) - \frac{1}{2}\inf_{\lambda\in\Omega}\left(\left\|A\widehat{\mathscr{K}_\lambda}\right\| - \left\|A^*\widehat{\mathscr{K}_\lambda}\right\|\right)^2.$$

Proof. Let $\lambda \in \Omega$ be arbitrary. By applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|\widetilde{A}(\lambda)\right|^2 + \left\|A\widehat{\mathscr{K}_\lambda}\right\|^4 \\
&= \left|\widetilde{A}(\lambda)\right|^2 + \left(\widetilde{A^*A}(\lambda)\right)^2 \\
&\leqslant \widetilde{|A|^2}^{1/2}(\lambda)\,\widetilde{|A^*|^2}^{\frac{1}{2}} + \widetilde{|A^*A|^2}^{\frac{1}{2}}\,\widetilde{|A^*A|^2}^{\frac{1}{2}} \\
&= \frac{1}{2}\left(\widetilde{|A|^2}(\lambda) + \widetilde{|A^*|^2}(\lambda) - \left(\widetilde{|A|^2}(\lambda)^{\frac{1}{2}} - \widetilde{|A^*|^2}(\lambda)^{\frac{1}{2}}\right)^2\right) + \widetilde{|A^*A|^2}(\lambda) \\
&= \frac{1}{2}\left(|A|^2 + 2|A|^4 + |A^*|^2\right)^{\sim}(\lambda) - \frac{1}{2}\left(\left\|A\widehat{\mathscr{K}_\lambda}\right\| - \left\|A^*\widehat{\mathscr{K}_\lambda}\right\|\right)^2 \\
&\leqslant \frac{1}{2}\mathrm{ber}\left(|A|^2 + 2|A|^4 + |A^*|^2\right) - \frac{1}{2}\inf_{\lambda\in\Omega}\left(\left\|A\widehat{\mathscr{K}_\lambda}\right\| - \left\|A^*\widehat{\mathscr{K}_\lambda}\right\|\right)^2.
\end{aligned}$$

Thus

$$\left|\widetilde{A}(\lambda)\right|^2 + \left\|A\widehat{\mathscr{K}_\lambda}\right\|^4 \leqslant \frac{1}{2}\mathrm{ber}\left(|A|^2 + 2|A|^4 + |A^*|^2\right) - \frac{1}{2}\inf_{\lambda\in\Omega}\left(\left\|A\widehat{\mathscr{K}_\lambda}\right\| - \left\|A^*\widehat{\mathscr{K}_\lambda}\right\|\right)^2.$$

Now the result follows by taking the supremum over all λ in Ω. □

To give our next result, we need Lemma 4 and the following lemma.

LEMMA 8. *Let $A \in \mathscr{B}(\mathscr{H})$. Then:*
(i) $\left|\widetilde{A}(\lambda)\right|^2 \leqslant \frac{1}{2}\left|\widetilde{A^2}(\lambda)\right| + \frac{1}{4}\left(|A|^2+|A^*|^2\right)^{\sim}(\lambda)$ *for all* $\lambda \in \Omega$*;*
(ii) $\operatorname{ber}^2(A) \leqslant \frac{1}{2}\operatorname{ber}\left(A^2\right) + \frac{1}{4}\left(\operatorname{ber}\left(|A|^2\right)+\operatorname{ber}\left(|A^*|^2\right)\right)$.

Proof. Let $\lambda \in \Omega$. Applying Lemma 4 for $a = A\widehat{\mathscr{K}_\lambda}$, $c = \widehat{\mathscr{K}_\lambda}$ and $b = A^*\widehat{\mathscr{K}_\lambda}$, we have that

$$\begin{aligned}
\left|\widetilde{A}(\lambda)\right|^2 &= \left|\left\langle A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 \\
&= \left|\left\langle A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\left\langle \widehat{\mathscr{K}_\lambda}, A^*\widehat{\mathscr{K}_\lambda}\right\rangle\right| \\
&\leqslant \frac{1}{2}\left(\left|\left\langle A\widehat{\mathscr{K}_\lambda}, A^*\widehat{\mathscr{K}_\lambda}\right\rangle\right| + \left\|A\widehat{\mathscr{K}_\lambda}\right\|\left\|A^*\widehat{\mathscr{K}_\lambda}\right\|\right) \\
&\leqslant \frac{1}{2}\left|\left\langle A\widehat{\mathscr{K}_\lambda}, A^*\widehat{\mathscr{K}_\lambda}\right\rangle\right| + \frac{1}{4}\left(\left\|A\widehat{\mathscr{K}_\lambda}\right\|^2 + \left\|A^*\widehat{\mathscr{K}_\lambda}\right\|^2\right) \\
&\quad \text{(by the arithmetic-geometric mean in equality)} \\
&= \frac{1}{2}\left|\left\langle A^2\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right| + \frac{1}{4}\left\langle\left(|A|^2+|A^*|^2\right)\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle.
\end{aligned}$$

This proves (i). The proof of (ii) is immediate from (i) by taking the supremum over all $\lambda \in \Omega$. The lemma is proved. □

We now prove the following result.

THEOREM 3. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$\eta^2(A) \leqslant \frac{1}{4}\left[\operatorname{ber}\left(\left(|A|^2+A\right)^2\right) + \operatorname{ber}\left(\left(|A|^2-A\right)^2\right) + \operatorname{ber}\left(|A|^2+2|A|^4+|A^*|^2\right)\right].$$

Proof. Let $\lambda \in \Omega$ be arbitrary. By the parallelogram identity for complex numbers we have :

$$\begin{aligned}
&\left|\widetilde{A}(\lambda)\right|^2 + \left\|A\widehat{\mathscr{K}_\lambda}\right\|^4 \\
&= \left|\left\langle A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 + \left\|A\widehat{\mathscr{K}_\lambda}\right\|^4 \\
&= \frac{1}{2}\left(\left|\left\|A\widehat{\mathscr{K}_\lambda}\right\|^2 + \left\langle A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 + \left\|A\widehat{\mathscr{K}_\lambda}\right\|^2\right) \\
&= \frac{1}{2}\left(\left|\left\langle\left(|A|^2+A\right)\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 + \left|\left\langle\left(|A|^2-A\right)\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|^2\right) \\
&\leqslant \frac{1}{2}\left[\frac{1}{2}\left|\left\langle\left(|A|^2+A\right)^2\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|\right.
\end{aligned}$$

$$+\frac{1}{4}\left\langle\left(\left||A|^2+A\right|^2+\left||A|^2+A^*\right|^2\right)\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle$$
$$+\frac{1}{2}\left|\left\langle\left(|A|^2-A\right)^2\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|+\frac{1}{4}\left\langle\left(\left||A|^2-A\right|^2+\left||A|^2-A^*\right|^2\right)\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\Bigg]$$

(by Lemma 8)

$$=\frac{1}{4}\left|\left\langle\left(|A|^2+A\right)^2\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|+\frac{1}{4}\left|\left\langle\left(|A|^2-A\right)^2\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|$$
$$+\frac{1}{4}\left\langle\left(|A|^2+2|A|^4+|A^*|^2\right)\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle$$
$$\leqslant\frac{1}{4}\left[\mathrm{ber}\left(\left(|A|^2+A\right)^2\right)+\mathrm{ber}\left(\left(|A|^2-A\right)^2\right)+\mathrm{ber}\left(|A|^2+2|A|^4+|A^*|^2\right)\right].$$

Thus

$$\left|\widetilde{A}(\lambda)\right|^2+\left\|A\widehat{\mathscr{K}_\lambda}\right\|^4$$
$$\leqslant\frac{1}{4}\left[\mathrm{ber}\left(\left(|A|^2+A\right)^2\right)+\mathrm{ber}\left(\left(|A|^2-A\right)^2\right)+\mathrm{ber}\left(|A|^2+2|A|^4+|A^*|^2\right)\right].$$

for all $\lambda\in\Omega$, which implies by taking the supremum over $\lambda\in\Omega$ the desired inequality. □

Our next result gives another upper bound for the Davis-Wielandt-Berezin radius of reproducing kernel Hilbert space operators.

THEOREM 4. *Let $A\in\mathscr{B}(\mathscr{H})$. Then:*
(i) $\mathrm{ber}^2(A)\leqslant\frac{1}{2}\mathrm{ber}\left(A^2\right)+\frac{1}{4}\mathrm{ber}\left(|A|^2+|A^*|^2\right)$;
(ii)

$$\eta^2(A)\leqslant\frac{1}{2}\mathrm{ber}\left(A^2\right)+\frac{1}{4}\mathrm{ber}\left(|A|^2+|A^*|^2\right)$$
$$+4\mathrm{ber}^2(A)\left(2\mathrm{ber}^2(A)-\widetilde{c}^2(A)+2\mathrm{ber}(A)\sqrt{\mathrm{ber}^2(A)-\widetilde{c}^2(A)}\right).\tag{4}$$

Proof. Let $\lambda\in\Omega$ be an arbitrary point.
(i) It follows from Lemma 8 (i) that

$$\left|\widetilde{A}(\lambda)\right|^2=\left|\left\langle A\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2$$
$$\leqslant\frac{1}{2}\left|\left\langle A^2\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|+\frac{1}{4}\left\langle\left(|A|^2+|A^*|^2\right)\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle$$
$$=\frac{1}{2}\left|\widetilde{A^2}(\lambda)\right|+\frac{1}{4}\left(\widetilde{|A|^2+|A^*|^2}(\lambda)\right).\tag{5}$$

Hence

$$\sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|^2\leqslant\frac{1}{2}\sup_{\lambda\in\Omega}\left|\widetilde{A^2}(\lambda)\right|^2+\frac{1}{4}\sup_{\lambda\in\Omega}\left(\widetilde{|A|^2+|A^*|^2}(\lambda)\right),$$

that is

$$\operatorname{ber}^2(A) \leqslant \frac{1}{2}\operatorname{ber}\left(A^2\right) + \frac{1}{4}\operatorname{ber}\left(|A|^2 + |A^*|^2\right),$$

which proves (i).

(ii) By Lemma 5 we get

$$\begin{aligned}
\left\|A\widehat{\mathscr{K}_\lambda}\right\|^4 &\leqslant 16\left(\frac{\operatorname{ber}^2(A)}{2} + \frac{\operatorname{ber}(A)}{2}\sqrt{\operatorname{ber}^2(A) - \left|\left\langle A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|}\right)^2 \\
&\leqslant 4\left(\operatorname{ber}^2(A) + \operatorname{ber}(A)\sqrt{\operatorname{ber}^2(A) - \widetilde{c}^2(A)}\right) \\
&\leqslant 4\operatorname{ber}^2(A)\left(2\operatorname{ber}^2(A) - \widetilde{c}^2(A) + 2\operatorname{ber}(A)\sqrt{\operatorname{ber}^2(A) - \widetilde{c}^2(A)}\right),
\end{aligned}$$

and hence

$$\left\|A\widehat{\mathscr{K}_\lambda}\right\|^4 \leqslant 4\operatorname{ber}^2(A)\left(2\operatorname{ber}^2(A) - \widetilde{c}^2(A) + 2\operatorname{ber}(A)\sqrt{\operatorname{ber}^2(A) - \widetilde{c}^2(A)}\right).$$

By (5) and (6) we obtain that

$$\begin{aligned}
&\left|\left\langle A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 + \left\|A\widehat{\mathscr{K}_\lambda}\right\|^4 \\
&\leqslant \frac{1}{2}\operatorname{ber}\left(A^2\right) + \frac{1}{4}\operatorname{ber}\left(|A|^2 + |A^*|^2\right) \\
&\quad + 4\operatorname{ber}^2(A)\left(2\operatorname{ber}^2(A) - \widetilde{c}^2(A) + 2\operatorname{ber}(A)\sqrt{\operatorname{ber}^2(A) - \widetilde{c}^2(A)}\right),
\end{aligned}$$

hence, on taking the supremum in this inequality over $\lambda \in \Omega$, we obtain the desired inequality (4) for the Davis-Wielandt-Berezin radius. The theorem is proved. □

In the sequel, we need the following two lemmas.

LEMMA 9. *For any* $\alpha, \beta \in \mathbb{C}$

$$\sup_{|\zeta|^2 + |\xi|^2 \leqslant 1} |\zeta\alpha + \xi\beta|^2 = |\alpha|^2 + |\beta|^2.$$

Proof. The proof is trivial. □

LEMMA 10. *Let* $A, B \in \mathscr{B}(\mathscr{H})$ *and* $\zeta, \xi \in \mathbb{C}$. *Then:*

(i) $\|\zeta A + \xi B\|_{Ber}^2 \leqslant \left(|\zeta|^2 + |\xi|^2\right)\operatorname{ber}\left(|A|^2 + |B|^2\right)$.

(ii) $\|\zeta A + \xi B\|^2 \leqslant \left(|\zeta|^2 + |\xi|^2\right)\|A^*A + B^*B\|$.

Proof. (i) Due to the Cauchy-Bunyakovsky-Schwarz inequality, for any $\lambda \in \Omega$, we have

$$\begin{aligned}\left\|(\zeta A+\xi B)\widehat{\mathscr{K}_\lambda}\right\|^2 &\leqslant \left(|\zeta|^2+|\xi|^2\right)\left(\left\|A\widehat{\mathscr{K}_\lambda}\right\|^2+\left\|B\widehat{\mathscr{K}_\lambda}\right\|^2\right)\\ &= \left(|\zeta|^2+|\xi|^2\right)\left\langle (A^*A+B^*B)\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\\ &\leqslant \left(|\zeta|^2+|\xi|^2\right)\operatorname{ber}(A^*A+B^*B),\end{aligned}$$

and so

$$\sup_{\lambda\in\Omega}\left\|(\zeta A+\xi B)\widehat{\mathscr{K}_\lambda}\right\|^2 \leqslant \left(|\zeta|^2+|\xi|^2\right)\operatorname{ber}\left(|A|^2+|B|^2\right),$$

which means that

$$\left\|(\zeta A+\xi B)\widehat{\mathscr{K}_\lambda}\right\|_{Ber}^2 \leqslant \left(|\zeta|^2+|\xi|^2\right)\operatorname{ber}\left(|A|^2+|B|^2\right),$$

as desired.

(ii) See the proof of Lemma 2.10 in [34]. □

We are now in a position to prove our main result.

THEOREM 5. *If $A \in \mathscr{B}(\mathscr{H})$, then*

(i) $\eta(A) \leqslant \left(\left\||A|^2\right\|_{Ber}+\left\||A|^4\right\|_{Ber}+2\operatorname{ber}^2\left(|A|^2A\right)\right)^{1/4}$;

(ii) $\eta(A) \leqslant \left[w\left(|A|^4+|A|^8\right)+2\operatorname{ber}^2\left(|A|^2A\right)\right]^{1/4}$.

Proof. Let $\lambda \in \Omega$ be any point.

(i) Putting in Lemma 6 $a=\widehat{\mathscr{K}_\lambda}, b=A\widehat{\mathscr{K}_\lambda}$ and $c=|A|^2\widehat{\mathscr{K}_\lambda}$, we get

$$\begin{aligned}&\left(\left|\widetilde{A}(\lambda)\right|^2+\left\|A\widehat{\mathscr{K}_\lambda}\right\|^4\right)^2\\ &=\left(\left|\left\langle\widehat{\mathscr{K}_\lambda},A\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2+\left|\left\langle\widehat{\mathscr{K}_\lambda},|A|^2\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2\right)^2\\ &\leqslant\left\|\widehat{\mathscr{K}_\lambda}\right\|^4\left(\left|\left\langle A\widehat{\mathscr{K}_\lambda},A\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2+2\left|\left\langle A\widehat{\mathscr{K}_\lambda},|A|^2\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2+\left|\left\langle |A|^2\widehat{\mathscr{K}_\lambda},|A|^2\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2\right)\\ &=\left|\left\langle |A|^2\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2+\left|\left\langle |A|^4\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2+2\left|\left\langle |A|^2A\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2\\ &\leqslant\sup_{\lambda\in\Omega}\left(\left|\left\langle |A|^2\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2+\left|\left\langle |A|^4\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2\right)+2\operatorname{ber}^2\left(|A|^2A\right)\\ &=\sup_{\lambda\in\Omega}\left(\sup_{|\zeta|^2+|\xi|^2\leqslant 1}\left|\zeta\left\langle |A|^2\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle+\xi\left\langle |A|^4\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2\right)+2\operatorname{ber}^2\left(|A|^2A\right)\end{aligned}$$

(by Lemma 9)

$$= \sup_{\lambda\in\Omega}\left(\sup_{|\zeta|^2+|\xi|^2\leqslant 1}\left|\left\langle\left(\zeta|A|^2+\xi|A|^4\right)\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2\right)+2\mathrm{ber}^2\left(|A|^2A\right)$$
$$= \sup_{|\zeta|^2+|\xi|^2\leqslant 1}\left(\sup_{\lambda\in\Omega}\left|\left\langle\left(\zeta|A|^2+\xi|A|^4\right)\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2\right)+2\mathrm{ber}^2\left(|A|^2A\right)$$
$$\leqslant \left\||A|^2\right\|_{Ber}+\left\||A|^4\right\|_{Ber}+2\mathrm{ber}^2\left(|A|^2A\right).$$

Hence,

$$\eta(A)=\sup_{\lambda\in\Omega}\left(\left|\widetilde{A}(\lambda)\right|^2+\left\|A\widehat{\mathscr{K}_\lambda}\right\|^4\right)^2\leqslant\left[\left\||A|^2\right\|_{Ber}+\left\||A|^4\right\|_{Ber}+2\mathrm{ber}^2\left(|A|^2A\right)\right]^{1/4},$$

as desired to prove.

(ii) Recall that for any normal operator A on a Hilbert space H its norm coincides with the numerical radius. Using this fact the inequality $\mathrm{ber}(A)\leqslant w(A)$ and the above established inequality

$$\left(\left|\widetilde{A}(\lambda)\right|^2+\left\|A\widehat{\mathscr{K}_\lambda}\right\|^4\right)^2\leqslant\sup_{|\zeta|^2+|\xi|^2\leqslant 1}\left(\sup_{\lambda\in\Omega}\left|\left\langle\left(\zeta|A|^2+\xi|A|^4\right)\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2\right)$$
$$+2\mathrm{ber}^2\left(|A|^2A\right)$$

we get

$$\left(\left|\widetilde{A}(\lambda)\right|^2+\left\|A\widehat{\mathscr{K}_\lambda}\right\|^4\right)^2$$
$$\leqslant\sup_{|\zeta|^2+|\xi|^2\leqslant 1}\left(\sup_{\lambda\in\Omega}\left|\left\langle\left(\zeta|A|^2+\xi|A|^4\right)\widehat{\mathscr{K}_\lambda},\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2\right)+2\mathrm{ber}^2\left(|A|^2A\right)$$
$$\leqslant\sup_{|\zeta|^2+|\xi|^2\leqslant 1}w^2\left(\zeta|A|^2+\xi|A|^4\right)+2\mathrm{ber}^2\left(|A|^2A\right)$$
$$\leqslant\sup_{|\zeta|^2+|\xi|^2\leqslant 1}\left\|\zeta|A|^2+\xi|A|^4\right\|^2+2\mathrm{ber}^2\left(|A|^2A\right)$$
$$\leqslant\sup_{|\zeta|^2+|\xi|^2\leqslant 1}\left(|\zeta|^2+|\xi|^2\right)\left\|\left(|A|^2\right)^*|A|^2+\left(|A|^4\right)^*|A|^4\right\|+2\mathrm{ber}^2\left(|A|^2A\right)$$

(by Lemma 10 (ii))

$$=\left\||A|^4+|A|^8\right\|+2\mathrm{ber}^2\left(|A|^2A\right)=w\left(|A|^4+|A|^8\right)+2\mathrm{ber}^2\left(|A|^2A\right)$$

(since $|A|^4+|A|^8$ is a normal operator).

Hence

$$\left(\left|\widetilde{A}(\lambda)\right|^2+\left\|A\widehat{\mathscr{K}_\lambda}\right\|^4\right)^{1/2}\leqslant\left[w\left(|A|^4+|A|^8\right)+2\mathrm{ber}^2\left(|A|^2A\right)\right]^{1/4}$$

for all $\lambda \in \Omega$, and consequently

$$\eta(A) \leqslant \left[w\left(|A|^4 + |A|^8\right) + 2\mathrm{ber}^2\left(|A|^2 A\right)\right]^{1/4},$$

which proves (ii). The theorem is proved. □

Next upper bound for the Davis-Wielandt-Berezin radius of operators can be stated as follows.

PROPOSITION 1. *Let $A \in \mathscr{B}(\mathscr{H})$. Then*

$$\eta(A) \leqslant \left[\max\left\{\|A\|_{\mathrm{Ber}}^2, \left\||A|^2\right\|_{\mathrm{Ber}}^2\right\} + \sqrt{2}\mathrm{ber}\left(|A|^2 A\right)\right]^{1/2}.$$

Proof. Let $\lambda \in \Omega$ be any point. Choosing in Lemma 7 $a = \widehat{\mathscr{K}_\lambda}, b = A\widehat{\mathscr{K}_\lambda}$ and $c = |A|^2 \widehat{\mathscr{K}_\lambda}$, we have

$$\begin{aligned}
&\left|\left\langle A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 + \left\|A\widehat{\mathscr{K}_\lambda}\right\|^4 \left|\left\langle \widehat{\mathscr{K}_\lambda}, A\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 + \left|\left\langle \widehat{\mathscr{K}_\lambda}, A^*A\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 \\
&= \left|\left\langle \widehat{\mathscr{K}_\lambda}, A\widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 + \left|\left\langle \widehat{\mathscr{K}_\lambda}, |A|^2 \widehat{\mathscr{K}_\lambda}\right\rangle\right|^2 \\
&\leqslant \left\|A\widehat{\mathscr{K}_\lambda}\right\|^2 + \left\||A|^2 \widehat{\mathscr{K}_\lambda}\right\|^2 \\
&\leqslant \max\left\{\left\|A\widehat{\mathscr{K}_\lambda}\right\|^2, \left\||A|^2 \widehat{\mathscr{K}_\lambda}\right\|^2\right\} + \sqrt{2}\left|\left\langle A\widehat{\mathscr{K}_\lambda}, |A|^2 \widehat{\mathscr{K}_\lambda}\right\rangle\right| \\
&\leqslant \max\left\{\|A\|_{\mathrm{Ber}}^2, \left\||A|^2\right\|_{\mathrm{Ber}}^2\right\} + \sqrt{2}\left|\left\langle |A|^2 A\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle\right| \\
&\leqslant \max\left\{\|A\|_{\mathrm{Ber}}^2, \left\||A|^2\right\|_{\mathrm{Ber}}^2\right\} + \sqrt{2}\mathrm{ber}\left(|A|^2 A\right),
\end{aligned}$$

and hence

$$\sup_{\lambda \in \Omega}\left(\left|\widetilde{A}(\lambda)\right|^2 + \left\|A\widehat{\mathscr{K}_\lambda}\right\|^4\right)^{1/2} \leqslant \left[\max\left\{\|A\|_{\mathrm{Ber}}^2, \left\||A|^2\right\|_{\mathrm{Ber}}^2\right\} + \sqrt{2}\mathrm{ber}\left(|A|^2 A\right)\right]^{1/2},$$

as desired. □

3.2. Berezin number inequalities for quadratic weighted operator means

In this subsection, we prove some Berezin number inequalities for the quadratic weighted operator geometric mean of (A_1, A_2), where A_1 and A_2 have similar positive parts.

THEOREM 6. *Let $A_1 \in \mathscr{B}^{-1}(\mathscr{H})$ and $A_2 \in \mathscr{B}(\mathscr{H})$ be two positive operators such that A_1 and A_2 have similar positive parts i.e., there exists an unitary operator $U : \mathscr{H} \to \mathscr{H}$ such that $|A_1| = U^* |A_2| U$. Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\operatorname{ber}\left(A_1 Ⓢ_{1/p} A_2\right) \leqslant \frac{1}{p} \|A_2\|_{\mathrm{Ber}}^2 + \frac{1}{q} \|A_2 U\|_{\mathrm{Ber}}^2 .$$

Proof. Let $A_1 = V_1 |A_1|$ and $A_2 = V_2 |A_2|$ be the polar decomposition of A_1 and A_2, respectively. Since A_1 and A_2 have similar positive parts, we have

$$|A_1| = U^* |A_2| U$$

and

$$|A_1|^2 = U^* |A_2|^2 U.$$

According to Lemma 3, we obtain

$$\left\langle A_1 Ⓢ_{1/p} A_2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda} \right\rangle \leqslant \left\langle |A_2|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda} \right\rangle^{1/p} \left\langle |A_1|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda} \right\rangle^{1/q},$$

which is less than equal to

$$\frac{1}{p} \left\langle |A_2|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda} \right\rangle + \frac{1}{q} \left\langle |A_1|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda} \right\rangle$$

by the classical inequality

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$$

for $a, b \geqslant 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ (see, for instance, Hardy, Littlewood, Polya [18]). This is equal to

$$\frac{1}{p} \left\langle |A_2|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda} \right\rangle + \frac{1}{q} \left\langle U^* |A_2|^2 U \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda} \right\rangle$$

as A_1 and A_2 have similar positive parts. Then we have for any $\lambda \in \Omega$ that

$$\begin{aligned}
\widetilde{A_1 Ⓢ_{1/p} A_2}(\lambda) &= \left\langle A_1 Ⓢ_{1/p} A_2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda} \right\rangle \\
&\leqslant \frac{1}{p} \left\langle |A_2|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda} \right\rangle + \frac{1}{q} \left\langle |A_2|^2 U \widehat{\mathscr{K}_\lambda}, U \widehat{\mathscr{K}_\lambda} \right\rangle \\
&= \frac{1}{p} \left\langle |A_2| \widehat{\mathscr{K}_\lambda}, |A_2| \widehat{\mathscr{K}_\lambda} \right\rangle + \frac{1}{q} \left\langle |A_2| U \widehat{\mathscr{K}_\lambda}, |A_2| U \widehat{\mathscr{K}_\lambda} \right\rangle \\
&= \frac{1}{p} \left\langle V_2^* V_2 |A_2| \widehat{\mathscr{K}_\lambda}, |A_2| \widehat{\mathscr{K}_\lambda} \right\rangle + \frac{1}{q} \left\langle V_2^* V_2 |A_2| U \widehat{\mathscr{K}_\lambda}, |A_2| U \widehat{\mathscr{K}_\lambda} \right\rangle \\
&= \frac{1}{p} \left\langle V_2 |A_2| \widehat{\mathscr{K}_\lambda}, V_2 |A_2| \widehat{\mathscr{K}_\lambda} \right\rangle + \frac{1}{q} \left\langle V_2 |A_2| U \widehat{\mathscr{K}_\lambda}, V_2 |A_2| U \widehat{\mathscr{K}_\lambda} \right\rangle \\
&= \frac{1}{p} \left\langle A_2 \widehat{\mathscr{K}_\lambda}, A_2 \widehat{\mathscr{K}_\lambda} \right\rangle + \frac{1}{q} \left\langle A_2 U \widehat{\mathscr{K}_\lambda}, A_2 U \widehat{\mathscr{K}_\lambda} \right\rangle \\
&= \frac{1}{p} \left\| A_2 \widehat{\mathscr{K}_\lambda} \right\|^2 + \frac{1}{q} \left\| A_2 U \widehat{\mathscr{K}_\lambda} \right\|^2 .
\end{aligned}$$

Taking supremum over all $\lambda \in \Omega$, we thus have

$$\operatorname{ber}\left(A_1 \circledS_{1/p} A_2\right) \leqslant \frac{1}{p}\left\|A_2\right\|_{\mathrm{Ber}}^2+\frac{1}{q}\left\|A_2 U\right\|_{\mathrm{Ber}}^2,$$

as desired. □

COROLLARY 1. *Let $A_1 \in \mathscr{B}^{-1}(\mathscr{H})$ and $A_2 \in \mathscr{B}(\mathscr{H})$ be two positive operators, and let $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\begin{aligned}\operatorname{ber}\left(A_1 \circledS_{1/p} A_2\right) \leqslant & \frac{1}{p} \operatorname{ber}\left(\left|A_2\right|^2\right)+\frac{1}{q} \operatorname{ber}\left(\left|A_1\right|^2\right) \\ & -\min \left\{\frac{1}{p}, \frac{1}{q}\right\} \inf _{\lambda \in \Omega}\left[\left(\widetilde{\left|A_2\right|^2}(\lambda)\right)^{1/2}-\left(\widetilde{\left|A_1\right|^2}(\lambda)\right)^{1/2}\right]^2 .\end{aligned}$$

Proof. Indeed, by Lemma 3, we have that

$$\begin{aligned}
& \left\langle A_1 \circledS_{1/p} A_2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle \\
& \leqslant\left\langle\left|A_2\right|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle^{1/p}\left\langle\left|A_1\right|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle^{1/q} \\
& \leqslant \frac{1}{p}\left\langle\left|A_2\right|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle+\frac{1}{q}\left\langle\left|A_1\right|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle \\
& \quad-\min \left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\left\langle\left|A_2\right|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2}-\left\langle\left|A_1\right|^2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2}\right)^2 \text{(by Lemma 2).}
\end{aligned}$$

Then

$$\begin{aligned}
& \left\langle A_1 \circledS_{1/p} A_2 \widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle \\
& \leqslant \frac{1}{p}\left\langle\left|A_2\right| \widehat{\mathscr{K}_\lambda},\left|A_2\right| \widehat{\mathscr{K}_\lambda}\right\rangle+\frac{1}{q}\left\langle\left|A_1\right| \widehat{\mathscr{K}_\lambda},\left|A_1\right| \widehat{\mathscr{K}_\lambda}\right\rangle \\
& \quad-\min \left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\left\langle\left|A_2\right| \widehat{\mathscr{K}_\lambda},\left|A_2\right| \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2}-\left\langle\left|A_1\right| \widehat{\mathscr{K}_\lambda},\left|A_1\right| \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2}\right)^2 \\
& =\frac{1}{p}\left\langle V_2^* V_2\left|A_2\right| \widehat{\mathscr{K}_\lambda},\left|A_2\right| \widehat{\mathscr{K}_\lambda}\right\rangle+\frac{1}{q}\left\langle V_1^* V_1\left|A_1\right| \widehat{\mathscr{K}_\lambda},\left|A_1\right| \widehat{\mathscr{K}_\lambda}\right\rangle \\
& \quad-\min \left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\left\langle V_2^* V_2\left|A_2\right| \widehat{\mathscr{K}_\lambda},\left|A_2\right| \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2}-\left\langle V_1^* V_1\left|A_1\right| \widehat{\mathscr{K}_\lambda},\left|A_1\right| \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2}\right)^2 \\
& =\frac{1}{p}\left\langle V_2\left|A_2\right| \widehat{\mathscr{K}_\lambda}, V_2\left|A_2\right| \widehat{\mathscr{K}_\lambda}\right\rangle+\frac{1}{q}\left\langle V_1\left|A_1\right| \widehat{\mathscr{K}_\lambda}, V_1\left|A_1\right| \widehat{\mathscr{K}_\lambda}\right\rangle \\
& \quad-\min \left\{\frac{1}{p}, \frac{1}{q}\right\}\left(\left\langle V_2\left|A_2\right| \widehat{\mathscr{K}_\lambda}, V_2\left|A_2\right| \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2}+\left\langle V_1\left|A_1\right| \widehat{\mathscr{K}_\lambda}, V_1\left|A_1\right| \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p}\left\langle A_2\widehat{\mathscr{K}_\lambda}, A_2\widehat{\mathscr{K}_\lambda}\right\rangle + \frac{1}{q}\left\langle A_1\widehat{\mathscr{K}_\lambda}, A_1\widehat{\mathscr{K}_\lambda}\right\rangle \\
&\quad - \min\left\{\frac{1}{p},\frac{1}{q}\right\}\left(\left\langle A_2\widehat{\mathscr{K}_\lambda}, A_2\widehat{\mathscr{K}_\lambda}\right\rangle^{1/2} - \left\langle A_1\widehat{\mathscr{K}_\lambda}, A_1\widehat{\mathscr{K}_\lambda}\right\rangle^{1/2}\right)^2 \\
&= \frac{1}{p}\left\langle A_2^*A_2\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle + \frac{1}{q}\left\langle A_1^*A_1\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle \\
&\quad - \min\left\{\frac{1}{p},\frac{1}{q}\right\}\left(\left\langle A_2^*A_2\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2} - \left\langle A_1^*A_1\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2}\right)^2.
\end{aligned}
$$

Taking supremum over all $\lambda \in \Omega$, we thus have

$$
\begin{aligned}
&\mathrm{ber}\left(A_1 Ⓢ_{1/p} A_2\right) \\
&\leqslant \frac{1}{p}\mathrm{ber}\left(A_2^*A_2\right) + \frac{1}{q}\mathrm{ber}\left(A_1^*A_1\right) \\
&\quad - \min\left\{\frac{1}{p},\frac{1}{q}\right\}\inf_{\lambda\in\Omega}\left(\left\langle |A_2|^2\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2} - \left\langle |A_1|^2\widehat{\mathscr{K}_\lambda}, \widehat{\mathscr{K}_\lambda}\right\rangle^{1/2}\right)^2 \\
&= \frac{1}{p}\mathrm{ber}\left(|A_2|^2\right) + \frac{1}{q}\mathrm{ber}\left(|A_1|^2\right) \\
&\quad - \min\left\{\frac{1}{p},\frac{1}{q}\right\}\inf_{\lambda\in\Omega}\left[\left(\widetilde{|A_2|^2}(\lambda)\right)^{1/2} - \left(\widetilde{|A_1|^2}(\lambda)\right)^{1/2}\right]^2
\end{aligned}
$$

which completes the proof. □

For other results on Berezin number inequalities, see [1, 2, 11, 12, 13, 14, 16, 29, 30, 31].

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