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CHARACTERIZATIONS OF ELEMENTARY OPERATORS

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Abstract. Let $\mathscr{A}$ be an ultraprime algebra and $\mathscr{I}$ a closed ideal in $\mathscr{A}$ with left (resp. right) approximate unit. We characterize elementary operators on $\mathscr{A}$ in terms of their images. We show that if Φ is an elementary operator on $\mathscr{A}$, then the set $\Phi(\mathscr{A}_1)$ (where $\mathscr{A}_1$ is the unit ball of $\mathscr{A}$) is a left (resp. right) uniformly approximable subset of $\mathscr{I}$ if and only if for any minimal length representation $\sum_{i=1}^{k} M_{a_i,b_i}$ of Φ we have $\{a_i\}_{i=1}^{k} \subseteq \mathscr{I}$ (resp. $\{b_i\}_{i=1}^{k} \subseteq \mathscr{I}$).

1. Introduction

Let $\mathscr{A}$ be an algebra and $a,b \in \mathscr{A}$. The map $M_{a,b} : \mathscr{A} \to \mathscr{A}$ given by $M_{a,b}(x) = axb$ is called a *multiplication operator*. A map $\Phi : \mathscr{A} \to \mathscr{A}$ is called *elementary operator* if $\Phi = \sum_{i=1}^{n} M_{a_i,b_i}$ for $a_i, b_i \in \mathscr{A}$. The elements a_i, b_i, $i = 1,\dots,n$ are called the *symbols* of Φ. We denote by $\mathscr{EL}(\mathscr{A})$ the space of all elementary operators on $\mathscr{A}$ and by $\ell(\cdot)$ the minimal length of an elementary operator.

K. Vala in [13] proved that if X is a Banach space and $\mathscr{B}(X)$ is the algebra of all linear bounded maps on X, then the multiplication operator $M_{A,B} : \mathscr{B}(X) \to \mathscr{B}(X)$ is compact if and only if A and B are compact operators. Compact elementary operators on C^*-algebras have been studied by C. K. Fong and A. R. Sourour in [4], M. Mathieu in [7] and R. M. Timoney in [12].

Another topic considered in the literature is the following: If $\mathscr{A}$ is an algebra and $\mathscr{I}$ is an ideal of $\mathscr{A}$, characterize the elementary operators on $\mathscr{A}$, that map $\mathscr{A}$ into $\mathscr{I}$.

Let H be a Hilbert space, $\mathscr{B}(H)$ the algebra of all linear bounded maps on H and $\mathscr{K}(H)$ the algebra of all compact linear maps on H. C. K. Fong and A. R. Sourour have proved in [4] that if Φ is an elementary operator on $\mathscr{B}(H)$, then $\Phi(\mathscr{B}(H)) \subseteq \mathscr{K}(H)$ if and only if there exist $\{A_i\}_{i=1}^{k}, \{B_i\}_{i=1}^{k} \subseteq \mathscr{B}(H)$ such that at least one of A_i and B_i belongs to $\mathscr{K}(H)$ for $i = 1,\dots,k$ and $\Phi = \sum_{i=1}^{k} M_{A_i,B_i}$.

This result was generalized by M. Mathieu in the case where $\mathscr{A}$ is a normed algebra and $\mathscr{I}$ is an ultraprime ideal in $\mathscr{A}$ [9].

In this paper we study the following question:

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QUESTION 1.1. *Let $\mathscr{A}$ be a normed algebra, $\mathscr{I}$ an ideal of $\mathscr{A}$ and $\Phi = \sum_{i=1}^{k} M_{a_i,b_i}$ an elementary operator on $\mathscr{A}$ where $k = \ell(\Phi)$. Which conditions on the image of Φ yields the equivalence with any of the following?*

1. *all $a_i, b_j \in \mathscr{I}$,*
2. *all a_i or all $b_j \in \mathscr{I}$,*
3. *a_i or $b_i \in \mathscr{I}$ for each i.*

We provide an answer to this question for elementary operators on ultraprime normed algebras using the concept of uniformly approximable sets. The concept of uniformly approximable sets was introduced in [6] to study elementary operators on the algebra of the adjointable operators on a Hilbert module.

We also show that among the Banach algebras with approximate unit, the unital ones are those for which the unit ball is uniformly approximable.

A normed algebra $\mathscr{A}$ is said to be *ultraprime* if there is a constant $k_{\mathscr{A}} > 0$ such that $\|M_{a,b}\| \geqslant k_{\mathscr{A}} \|a\| \|b\|$ for all $a, b \in \mathscr{A}$, see [9]. The class of ultraprime normed algebras includes among others prime C^*-algebras [8, Proposition 2.3], $\mathscr{B}(X)$ where X is any normed space and prime group algebras $\ell^1(G)$ where G is a discrete group [14]. Moreover, ideals of ultraprime normed algebras are also ultraprime algebras [9, Proposition 3.6].

2. Characterizations of elementary operators

Let $\mathscr{A}$ be a normed algebra. By $\mathscr{A}_1$, we denote the closed unit ball of $\mathscr{A}$. By ideal we mean a closed ideal. A *left* (resp. *right*) *approximate unit* of $\mathscr{A}$ is a net $\{u_\lambda\}_{\lambda \in \Lambda}$ of elements of $\mathscr{A}$ such that:

1. for some positive number r, $\|u_\lambda\| \leqslant r$ for all $\lambda \in \Lambda$,
2. $\lim u_\lambda a = a$ (resp. $\lim a u_\lambda = a$) for all $a \in \mathscr{A}$ in the norm topology of $\mathscr{A}$.

A net which is both a left and a right approximate unit of $\mathscr{A}$ is called simply an *approximate unit* of $\mathscr{A}$.

The concept of uniformly approximable sets was introduced in [6] for subsets of C^*-algebras. We extend the definition for normed algebras with approximate unit.

DEFINITION 2.1. Let $\mathscr{A}$ be a normed algebra and $\mathscr{S}$ be a bounded subset of $\mathscr{A}$. The set $\mathscr{S}$ is called left (resp. right) uniformly approximable, if there is a left (resp. right) approximate unit $\{u_\lambda\}_{\lambda \in \Lambda}$ of $\mathscr{A}$ such that for every $\varepsilon > 0$ there exists $\lambda_0 \in \Lambda$ such that $\|u_\lambda s - s\| < \varepsilon$ (resp. $\|s u_\lambda - s\| < \varepsilon$) for all $\lambda \geqslant \lambda_0$ and for all $s \in \mathscr{S}$. A bounded subset $\mathscr{S}$ of $\mathscr{A}$ is called uniformly approximable, if there is an approximate unit $\{u_\lambda\}_{\lambda \in \Lambda}$ of $\mathscr{A}$ such that for every $\varepsilon > 0$ there exists $\lambda_0 \in \Lambda$ such that $\|u_\lambda s u_\lambda - s\| < \varepsilon$ for all $\lambda \geqslant \lambda_0$ and for all $s \in \mathscr{S}$.

If $\mathscr{S}$ is a uniformly approximable set with respect to an approximate unit $\{u_\lambda\}_{\lambda\in\Lambda}$ and $\{w_\mu\}_{\mu\in M}$ is another approximate unit, then $\mathscr{S}$ is also a uniformly approximable set with respect to w_μ and the analogous result holds for left approximable and right approximable sets. Hence, the definition of uniformly approximable sets does not depend on the choice of the approximate unit. The proof is similar to the proof of Lemma 4 in [6].

REMARK 2.2.

1. If $\mathscr{A}$ is a normed algebra with approximate unit and $\mathscr{S}$ is a uniformly approximable set, then $\mathscr{S}$ is left (resp. right) uniformly approximable set. This follows from the proof of Lemma 16 of [6]. On the other hand, if $\mathscr{S}$ is a left and right uniformly approximable set, applying the triangle inequality we see that $\mathscr{S}$ is a uniformly approximable set.

2. The following is an example of a set which is left uniformly approximable but is not right uniformly approximable. Let H be a separable infinite dimensional Hilbert space and $\mathscr{K}(H)$ be the algebra of compact linear operators on H. For $x,y\in H$ we denote by $x\otimes y$ the rank-one operator on H defined by $x\otimes y(z)=\langle z,y\rangle x$. If $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal sequence in H, then the subset

$$\mathscr{S}=\{e_1\otimes e_i : i\in\mathbb{N}\},$$

of $\mathscr{K}(H)$ is left uniformly approximable, but not right uniformly approximable.

If $\mathscr{A}$ is a unital normed algebra, then its unit ball is uniformly approximable. The following proposition shows that this property characterizes the unital Banach algebras among the Banach algebras with approximate unit.

PROPOSITION 2.3. *Let $\mathscr{A}$ be a Banach algebra with approximate unit. Then, the closed unit ball of $\mathscr{A}$ is a uniformly approximable set if and only if $\mathscr{A}$ is unital.*

Proof. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit of $\mathscr{A}$ and $r=\sup_\lambda\|u_\lambda\|$. We will prove that the net $\{u_\lambda\}_{\lambda\in\Lambda}$ is norm-convergent.

A uniformly approximable set is both left and right uniformly approximable set. Let $\varepsilon>0$. There is $\lambda'\in\Lambda$ such that $\|u_\lambda x-x\|\leqslant\frac{\varepsilon}{6r}$ for all $x\in\mathscr{A}_1$ and for all $\lambda>\lambda'$. Let $\lambda_0>\lambda'$. There is $\lambda''>\lambda_0$ such that $\|u_\lambda u_{\lambda_0}-u_{\lambda_0}\|\leqslant\frac{\varepsilon}{3}$ and $\|u_{\lambda_0}u_\lambda-u_{\lambda_0}\|\leqslant\frac{\varepsilon}{3}$ for all $\lambda>\lambda''$. Then, for any $\lambda_\mu,\lambda_\nu>\lambda''$ with $u_{\lambda_\mu}-u_{\lambda_\nu}\neq 0$ we have that:

$$\begin{aligned}
\frac{\varepsilon}{6r} &\geqslant \left\|u_{\lambda_0}\frac{u_{\lambda_\mu}-u_{\lambda_\nu}}{\|u_{\lambda_\mu}-u_{\lambda_\nu}\|}-\frac{u_{\lambda_\mu}-u_{\lambda_\nu}}{\|u_{\lambda_\mu}-u_{\lambda_\nu}\|}\right\| \\
&= \frac{1}{\|u_{\lambda_\mu}-u_{\lambda_\nu}\|}\left\|u_{\lambda_0}u_{\lambda_\mu}-u_{\lambda_0}u_{\lambda_\nu}-u_{\lambda_\mu}+u_{\lambda_\nu}\right\| \\
&\geqslant \frac{1}{2r}\left\|u_{\lambda_0}u_{\lambda_\mu}-u_{\lambda_0}+u_{\lambda_0}-u_{\lambda_0}u_{\lambda_\nu}-u_{\lambda_\mu}+u_{\lambda_\nu}\right\|
\end{aligned}$$

$$\geqslant \frac{1}{2r}\left(\|u_{\lambda_\mu}-u_{\lambda_\nu}\|-\|u_{\lambda_0}u_{\lambda_\mu}-u_{\lambda_0}\|-\|u_{\lambda_0}-u_{\lambda_0}u_{\lambda_\nu}\|\right)$$
$$\geqslant \frac{1}{2r}\left(\|u_{\lambda_\mu}-u_{\lambda_\nu}\|-\frac{2\varepsilon}{3}\right),$$

and hence $\|u_{\lambda_\mu}-u_{\lambda_\nu}\| \leqslant \varepsilon$. Therefore the net $\{u_\lambda\}_{\lambda\in\Lambda}$ has a limit in $\mathscr{A}$, which is a unit for $\mathscr{A}$. □

REMARK 2.4.

1. If $\mathscr{A}$ is a normed algebra with approximate unit and $\mathscr{S}$ is a totally bounded subset of $\mathscr{A}$, then $\mathscr{S}$ is a uniformly approximable set. Moreover, if the algebra $\mathscr{A}$ consists of compact elements, (an element $a\in\mathscr{A}$ is called compact if the operator $x\mapsto axa$ is a compact operator on $\mathscr{A}$), then a uniformly approximable set is totally bounded.

2. If $\mathscr{I}$ is an ideal with a left approximate unit in a normed algebra $\mathscr{A}$ and $a\in\mathscr{I}$, then the set $a\mathscr{A}_1$ is a left uniformly approximable subset of $\mathscr{I}$.

We shall use the following result which follows from [3, Theorem 3.1] and [9, Corollary 4.7] (see also [2, Theorem 10.1]). If $\mathscr{A}$ is an algebra without unit, we denote by $\mathscr{A}^1$ its unitization. If $\mathscr{A}$ is unital, $\mathscr{A}^1=\mathscr{A}$.

THEOREM 2.5. *Let $\mathscr{A}$ be an ultraprime normed algebra. If $b_1,\dots,b_n\in\mathscr{A}^1$, are such that b_1 does not lie in the linear span of $b_2,\dots,b_n$, then there exists an elementary operator $\Phi\in\mathscr{E}\mathscr{L}(\mathscr{A}^1)$ such that $\Phi(b_1)\neq 0$ and $\Phi(b_2)=\ldots=\Phi(b_n)=0$.*

If $\mathscr{I}$ is an ideal of a normed algebra $\mathscr{A}$, we denote $[.]_{\mathscr{A}/\mathscr{I}}:\mathscr{A}\to\mathscr{A}/\mathscr{I}$ the quotient map $x\mapsto[x]_{\mathscr{A}/\mathscr{I}}$. If $\mathscr{A}$ is a C^*-algebra, we denote $\mathscr{M}(\mathscr{A})$ the multiplier algebra of $\mathscr{A}$ [10, 2.1]. Recall that if $\mathscr{I}$ is an ideal of $\mathscr{A}$, then $\mathscr{I}$ is an ideal of $\mathscr{M}(\mathscr{A})$.

THEOREM 2.6. *Let $\mathscr{A}$ be a prime C^*-algebra, $\mathscr{I}$ be an ideal in $\mathscr{A}$ and $\Phi\in\mathscr{E}\mathscr{L}(\mathscr{A})$. If $\Phi=\sum_{i=1}^k M_{a_i,b_i}$ any minimal length representation of Φ where $\{a_i\}_{i=1}^k$, $\{b_i\}_{i=1}^k\subseteq\mathscr{M}(\mathscr{A})$, the following are equivalent:*

1. The set $\Phi(\mathscr{A}_1)$ is a left (resp. right) uniformly approximable subset of $\mathscr{I}$.

2. $\{a_i\}_{i=1}^k\subseteq\mathscr{I}$, (resp. $\{b_i\}_{i=1}^k\subseteq\mathscr{I}$).

Moreover, the set $\Phi(\mathscr{A}_1)$ is a uniformly approximable subset of $\mathscr{I}$ if and only if $\{a_i\}_{i=1}^k\subseteq\mathscr{I}$ and $\{b_i\}_{i=1}^k\subseteq\mathscr{I}$.

Proof. We show the implication $1\Rightarrow 2$. We assume that the set $\Phi(\mathscr{A}_1)$ is a left uniformly approximable subset of $\mathscr{I}$. Let $\Phi=\sum_{i=1}^k M_{a_i,b_i}$ be a minimal length representation of Φ such that $a_1\notin\mathscr{I}$. Since the representation is minimal, the elements of the set $\{b_1,\dots,b_k\}\subseteq\mathscr{M}(\mathscr{A})$ are linearly independent. It follows from Theorem 2.5, that there exists $\Psi=\sum_{i=1}^l M_{c_i,d_i}\in\mathscr{E}\mathscr{L}(\mathscr{M}(\mathscr{A}))$ such that $\Psi(b_1)\neq 0$ and $\Psi(b_i)=0$

for $i = 2,\ldots,k$. We assume that $\|c_i\|,\|d_i\| \leqslant 1$, for $i = 1,\ldots,l$. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit of $\mathscr{I}$. We consider the multiplication operator $M_{u'_\lambda a_1,\Psi(b_1)}$ on $\mathscr{A}$, where $u'_\lambda = 1 - u_\lambda \in \mathscr{M}(\mathscr{A})$. Since the algebra $\mathscr{A}$ is prime, it follows from [8, Proposition 2.3] that

$$\|M_{u'_\lambda a_1,\Psi(b_1)}\| = \|u'_\lambda a_1\|\|\Psi(b_1)\| \geqslant \|[a_1]_{\mathscr{M}/\mathscr{I}}\|\|\Psi(b_1)\| = r > 0,$$

for all $\lambda \in \Lambda$. Then, for $\lambda \in \Lambda$ there exists $y_\lambda \in \mathscr{A}_1$ such that

$$\|M_{u'_\lambda a_1,\Psi(b_1)}(y_\lambda)\| > r/2 > 0.$$

We observe that

$$\begin{aligned}\sum_{i=1}^{l}\Phi(y_\lambda c_i)d_i &= \sum_{i=1}^{l}\sum_{j=1}^{k} a_j y_\lambda c_i b_j d_i = \sum_{j=1}^{k}\sum_{i=1}^{l} a_j y_\lambda c_i b_j d_i \\ &= \sum_{j=1}^{k} a_j y_\lambda \sum_{i=1}^{l} c_i b_j d_i = \sum_{j=1}^{k} a_j y_\lambda \Psi(b_j) = a_1 y_\lambda \Psi(b_1),\end{aligned}$$

and hence

$$\left\|\sum_{i=1}^{l} u'_\lambda \Phi(y_\lambda c_i)d_i\right\| = \left\|u'_\lambda a_1 y_\lambda \Psi(b_1)\right\| > \frac{r}{2},$$

for all $\lambda \in \Lambda$. The set $\Phi(\mathscr{A}_1)$ is a left uniformly approximable subset of $\mathscr{I}$ and hence

$$\lim_\lambda \left\|\sum_{i=1}^{l} u'_\lambda \Phi(y_\lambda c_i)d_i\right\| = 0,$$

which is a contradiction.

The proof is similar in the case where the set $\Phi(\mathscr{A}_1)$ is a right uniformly approximable subset of $\mathscr{I}$. The implication $2 \Rightarrow 1$ follows taking into account Remark 2.4, (2).

The final assertion of the theorem follows from the equivalence of (1) and (2) and Remark 2.2 (1). □

REMARK 2.7.

1. Let $\mathscr{A}$ be a prime C^*-algebra and $\mathscr{K}(\mathscr{A})$ be the ideal of compact elements of $\mathscr{A}$. In [7, Corollary 3.9], Mathieu proved that an elementary operator $\Phi \in \mathscr{E}\mathscr{L}(\mathscr{A})$ is compact if and only if there are $\{a_i\}_{i=1}^k, \{b_i\}_{i=1}^k \subseteq \mathscr{K}(\mathscr{A})$ such that $\Phi = \sum_{i=1}^k M_{a_i,b_i}$. Hence, it follows from Theorem 2.6 that if $\mathscr{A}$ is a prime C^*-algebra an elementary operator Φ is compact if and only if the set $\Phi(\mathscr{A}_1)$ is a uniformly approximable subset of $\mathscr{K}(\mathscr{A})$.

2. Let $\mathscr{A}$ be a prime C^*-algebra and $\mathscr{K}(\mathscr{A})$ be the ideal of compact elements of $\mathscr{A}$. In [7, Theorem 3.7] Mathieu proves that an elementary operator $\Phi \in \mathscr{E}\mathscr{L}(\mathscr{A})$ is weakly compact if and only if there exist $\{a_i\}_{i=1}^k, \{b_i\}_{i=1}^k \subseteq \mathscr{A}$ such that at least one of tuples $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ lies in $\mathscr{K}(\mathscr{A})$ and $\Phi = \sum_{i=1}^k M_{a_i,b_i}$. It follows from Theorem 2.6 that if $\mathscr{A}$ is a prime C^*-algebra a multiplication operator Φ on $\mathscr{A}$ is weakly compact if and only if $\Phi(\mathscr{A}_1)$ is a left or right uniformly approximable subset of $\mathscr{K}(\mathscr{A})$.

3. If $\mathscr{A}$ is a C^*-algebra and $\mathscr{X}$ is a Hilbert $\mathscr{A}$-module we denote by $\mathscr{B}(\mathscr{X})$ the C^*-algebra of adjointable operators on $\mathscr{X}$ and by $\mathscr{K}(\mathscr{X})$ the C^*-subalgebra of $\mathscr{B}(\mathscr{X})$, consisting of the generalized compact operators on $\mathscr{X}$. In [6, Theorem 17] we proved that if $\mathscr{A}$ is a prime unital C^*-algebra, $\mathscr{X}$ is a countably generated Hilbert $\mathscr{A}$-module and Φ is an elementary operator on $\mathscr{B}(\mathscr{X})$, then the set $\Phi(\mathscr{B}(\mathscr{X})_1)$ is a uniformly approximable subset of $\mathscr{K}(\mathscr{X})$ if and only if there exist $\{a_i\}_{i=1}^k, \{b_i\}_{i=1}^k \subseteq \mathscr{K}(\mathscr{X})$ such that $\Phi = \sum_{i=1}^k M_{a_i,b_i}$. It follows from [1, Proposition 3.4], [5, Theorem 2.4] and [8, Lemma 2.2] that if the C^*-algebra $\mathscr{A}$ is prime, then the C^*-algebra $\mathscr{B}(\mathscr{X})$ is prime. Hence, it follows from the last assertion of Theorem 2.6, that the conclusion of [6, Theorem 17] holds without the assumption that $\mathscr{A}$ is unital or $\mathscr{X}$ is countably generated.

The proof of the following theorem is similar to the proof of Theorem 2.6 and is omitted. In contrast with our assumption in Theorem 2.6, the elementary operators considered have symbols in $\mathscr{A}$.

THEOREM 2.8. *Let $\mathscr{A}$ be an ultraprime normed algebra, $\mathscr{I}$ an ideal in $\mathscr{A}$ with left (resp. right) approximate unit and $\Phi \in \mathscr{E}\mathscr{L}(\mathscr{A})$. If $\Phi = \sum_{i=1}^k M_{a_i,b_i}$ is a minimal length representation of Φ where $\{a_i\}_{i=1}^k, \{b_i\}_{i=1}^k \subseteq \mathscr{A}$, the following are equivalent:*

1. *The set $\Phi(\mathscr{A}_1)$ is a left (resp. right) uniformly approximable subset of $\mathscr{I}$.*

2. *$\{a_i\}_{i=1}^k \subseteq \mathscr{I}$, (resp. $\{b_i\}_{i=1}^k \subseteq \mathscr{I}$).*

Moreover, the set $\Phi(\mathscr{A}_1)$ is a uniformly approximable subset of $\mathscr{I}$ if and only if $\{a_i\}_{i=1}^k \subseteq \mathscr{I}$ and $\{b_i\}_{i=1}^k \subseteq \mathscr{I}$.

A Banach algebra $\mathscr{A}$ is called hypocompact if any nonzero quotient of $\mathscr{A}$ by a closed ideal contains a nonzero compact element. A closed ideal $\mathscr{I}$ of $\mathscr{A}$ is called hypocompact if $\mathscr{I}$ is hypocompact as a Banach algebra. Shulman and Turovskii have proved that any Banach algebra $\mathscr{A}$ has a largest hypocompact ideal [11, Corollary 3.10]. We will denote by $\mathscr{A}_{hc}$ the largest hypocompact ideal of $\mathscr{A}$.

In [2, Corollary 10.6], Brešar and Turovskii proved that if $\mathscr{A}$ is a semisimple ultraprime Banach algebra and Φ is a compact elementary operator on $\mathscr{A}$, then there exist compact elements $\{a_i\}_{i=1}^k, \{b_i\}_{i=1}^k \subseteq \mathscr{A}^1$ such that $\Phi = \sum_{i=1}^k M_{a_i,b_i}$. In the following corollary of Theorem 2.8 we obtain an analogous result without the assumption of semisimplicity, but assuming instead that $\mathscr{A}_{hc}$ has an approximate unit.

COROLLARY 2.9. *Let $\mathscr{A}$ be an ultraprime Banach algebra such that the hypocompact radical $\mathscr{A}_{hc}$ has an approximate unit and $\Phi \in \mathscr{E}\mathscr{L}(\mathscr{A})$. If Φ is a compact elementary operator, then there exist $\{a_i\}_{i=1}^k, \{b_i\}_{i=1}^k \subseteq \mathscr{A}_{hc}$ such that $\Phi = \sum_{i=1}^k M_{a_i,b_i}$.*

Proof. It follows from [2, Theorem 8.4] that the range of Φ lies in $\mathscr{A}_{hc}$. Since Φ is compact, the set $\Phi(\mathscr{A}_1)$ is a uniformly approximable subset of $\mathscr{A}_{hc}$ (Remark 2.4 (1)). Hence, by Theorem 2.8 there exist $\{a_i\}_{i=1}^k, \{b_i\}_{i=1}^k \subseteq \mathscr{A}_{hc}$ such that $\Phi = \sum_{i=1}^k M_{a_i,b_i}$. □

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REFERENCES

[1] L. ARAMBAŠIĆ AND I. GOGIĆ, *Elementary operators on Hilbert modules over prime C^*-algebras*, J. Math. Anal. Appl. **485**, 2 (2020), 10 pp.
[2] M. BRESAR AND Y. V. TUROVSKII, *Compactness conditions for elementary operators*, Studia Math. **178**, 1 (2007), 1–18.
[3] T. S. ERICKSON, W. S. MARTINDALE 3RD AND J. M. OSBORN, *Prime nonassociative algebras*, Pacific J. Math. **60**, 1 (1975), 49–63.
[4] C. K. FONG AND A. R. SOUROUR, *On the operator identity $\sum A_k X B_k \equiv 0$*, Canad. J. Math. **31**, 4 (1979), 845–857.
[5] E. C. LANCE, *Hilbert C^*-Modules: A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series 210, Cambridge University Presss, Cambridge, 1995.
[6] C. MAGIATIS, *Elementary operators on the algebra of adjointable operators on a Hilbert module*, J. Math. Anal. Appl. **475**, 1 (2019), 628–640.
[7] M. MATHIEU, *Elementary operators on prime C^*-algebras, II*, Glasg. Math. J. **30**, 3 (1988), 275–284.
[8] M. MATHIEU, *Elementary operators on prime C^*-algebras, I*, Math. Ann. **284**, 2 (1989), 223–244.
[9] M. MATHIEU, *Rings of quotients of ultraprime Banach algebras, with applications to elementary operators*, in: Conference on Automatic Continuity and Banach Algebras (Canberra, 1989), Proc. Centre Math. Anal. Austral. Nat. Univ. **21**, Austral. Nat. Univ., Canberra, 1989, 297–317.
[10] G. J. MURPHY, *C^*-Algebras and Operator Theory*, Academic Press, Boston, 1990.
[11] V. S. SHULMAN AND YU. V. TUROVSKII, *Topological radicals and joint spectral radius*, Funktsional. Anal. i Prilozhen. **46**, 4 (2012), 61–82 (in Russian); English transl.: Funct. Anal. Appl. **46**, 4 (2012), 287–304.
[12] R. M. TIMONEY, *Some formulae for norms of elementary operators*, J. Operator Theory **57**, 1 (2007), 121–145.
[13] K. VALA, *On compact sets of compact operator*, Ann. Acad. Sci. Fenn. Ser. A I **351** (1964), 1–8.
[14] G. A. WILLIS, *Ultraprime group algebras*, Proc. Centre Math. Anal. Austral. Nat. Univ. **21** (1989), 345–349.

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