

## MAXIMAL NUMERICAL RANGE AND QUADRATIC ELEMENTS IN A $C^*$ -ALGEBRA

E. H. BENABDI, M. BARRAA, M. K. CHRAIBI AND A. BAGHDAD\*

*Dedicated to Professor A. Nokrane*

*(Communicated by I. M. Spitkovsky)*

*Abstract.* In this paper, we give a description of the maximal numerical range of a hyponormal element and a characterization of a normaloid element in a  $C^*$ -algebra. We also give an explicit formula for the maximal numerical range of a quadratic operator acting on a complex Hilbert space. As a consequence, we determine the maximal numerical range of a rank-one operator.

### 1. Introduction

Let  $\mathcal{A}$  be a complex  $C^*$ -algebra with unit  $e$  and let  $\mathcal{A}'$  be its dual space. Define the state space of  $\mathcal{A}$  by

$$\mathcal{S}(\mathcal{A}) = \{f \in \mathcal{A}' : f(e) = \|f\| = 1\}.$$

For  $a \in \mathcal{A}$ , the algebraic numerical range of  $a$  is given by

$$V(a) = \{f(a) : f \in \mathcal{S}(\mathcal{A})\}.$$

It is well-known that  $V(a)$  ( $a \in \mathcal{A}$ ) is a convex compact set and contains the convex hull of the spectrum  $\sigma(a)$  of  $a$ ; that is  $co(\sigma(a)) \subseteq V(a)$ , here  $co$  stands for the convex hull. This result follows at once from the corresponding properties of the set  $\mathcal{S}(\mathcal{A})$ . See, for more details, [16]. Let  $w(a)$  denote the numerical radius of  $a \in \mathcal{A}$ ; i.e.,  $w(a) = \sup\{|\lambda| : \lambda \in V(a)\}$ . It is well-known that  $w(\cdot)$  defines a norm on  $\mathcal{A}$ , which is equivalent to the  $C^*$ -norm  $\|\cdot\|$ . In fact, the following inequalities are well-known:

$$\frac{1}{2} \|a\| \leq w(a) \leq \|a\|,$$

for all  $a \in \mathcal{A}$ . An element  $a \in \mathcal{A}$  is said to be normaloid if  $w(a) = \|a\|$ . Recall that an element  $a \in \mathcal{A}$  is said to be positive and we write  $a \geq 0$  if it is self-adjoint and if its spectrum contains only non-negative real numbers. Recall also that an element  $a \in \mathcal{A}$

*Mathematics subject classification* (2020): Primary 46L05; Secondary 47A12, 47A63.

*Keywords and phrases:* Quadratic operator,  $C^*$ -algebra, numerical range, maximal numerical range.

\* Corresponding author.

is called normal (resp. hyponormal) if  $a^*a = aa^*$  (resp.  $a^*a - aa^* \geq 0$  or equivalently,  $a^*a - aa^* = b^*b$  for some  $b \in \mathcal{A}$ ). Here  $a^*$  is the adjoint of  $a$ . It is well-known that hyponormal, thus also normal, elements in  $\mathcal{A}$  are normaloid.

Let  $\mathcal{H}$  be a Hilbert space over the complex field  $\mathbb{C}$  with inner product  $\langle x, y \rangle$  and norm  $\|x\| = \langle x, x \rangle^{1/2}$ . Denote by  $\mathcal{B}(\mathcal{H})$  the  $C^*$ -algebra of all bounded linear operators acting on  $\mathcal{H}$ . For  $A \in \mathcal{B}(\mathcal{H})$ , the numerical range of  $A$  is defined as the set

$$W(A) = \{ \langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}.$$

It is a celebrated result due to Toeplitz-Hausdorff that  $W(A)$  is a convex set in the complex plane and it is well-known that  $\overline{W(A)} = V(A)$ , where  $\overline{L}$  is the closure of a subset  $L$  of  $\mathbb{C}$ . The numerical range of an operator in  $\mathcal{B}(\mathcal{H})$  is closed if  $\dim(\mathcal{H}) < \infty$ , but it is not always closed when  $\dim(\mathcal{H}) = \infty$ . For more details about the theory of numerical ranges, the reader is referred to [4, 5, 8, 9] and references therein.

The notion of the numerical range has been generalized in different directions. One such direction is the maximal numerical range. It is a relatively new concept in operator theory, having been introduced only in 1970 by Stampfli [17] and defined as follows.

DEFINITION 1.1. For  $A \in \mathcal{B}(\mathcal{H})$ , the maximal numerical range  $W_0(A)$  of  $A$  is given by

$$W_0(A) = \{ \lim_n \langle Ax_n, x_n \rangle : x_n \in \mathcal{H}, \|x_n\| = 1, \lim_n \|Ax_n\| = \|A\| \}.$$

It was shown in [17] that  $W_0(A)$  is nonempty, closed, convex and contained in the closure of the numerical range;  $W_0(A) \subseteq \overline{W(A)}$ . In the case of finite-dimensional spaces, the maximal numerical range is produced by maximal vectors for  $A$  (vectors  $x \in \mathcal{H}$  such that  $\|x\| = 1$  and  $\|Ax\| = \|A\|$ ). Note that the notion of the maximal numerical range was introduced by Stampfli [17] (especially) for the purpose of calculating the norm of the inner derivation on  $\mathcal{B}(\mathcal{H})$ . Recall that the inner derivation  $\delta_A$  associated with  $A \in \mathcal{B}(\mathcal{H})$  is defined by

$$\delta_A : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}), X \longmapsto AX - XA.$$

Indeed, the author of [17] established the following. For any  $A \in \mathcal{B}(\mathcal{H})$

$$\|\delta_A\| = 2 \|A - c_A\|,$$

where  $c_A$  is the unique scalar  $c_A$  satisfying

$$\|A - c_A\| = \inf_{\lambda \in \mathbb{C}} \|A - \lambda\|.$$

The scalar  $c_A$  is called the center of mass of  $A$ . In the same paper [17], Stampfli proved that we always have  $c_A \in \overline{W(A)}$ . Furthermore, if  $A$  is a hyponormal operator, the center of mass  $c_A$  is exactly the center of the smallest disk containing the spectrum  $\sigma(A)$ .

Recently, considerable interests have been given to the maximal numerical range, see, for instance, [3, 6, 10, 12, 15]. For example, in [3], the authors gave the following description of the maximal numerical range  $W_0(A)$  whenever  $A$  is hyponormal.

THEOREM 1.2. ([3]) *Let  $A \in \mathcal{B}(\mathcal{H})$  be hyponormal. Then*

$$W_0(A) = co(\sigma_n(A)),$$

where  $\sigma_n(A) := \{\lambda \in \sigma(A) : |\lambda| = \|A\|\}$ .

In [6] the authors gave the following characterization of normaloid operator.

THEOREM 1.3. ([6]) *Let  $A \in \mathcal{B}(\mathcal{H})$ . Then  $A$  is normaloid if and only if  $w(A) = w_0(A)$ , where  $w_0(a) := \sup\{|\lambda| : \lambda \in W_0(A)\}$ , the maximal numerical radius of  $A$ .*

In [7], the author introduced the concept of the algebraic maximal numerical range of an element  $a \in \mathcal{A}$  as follows.

DEFINITION 1.4. Let  $a \in \mathcal{A}$ . The algebraic maximal numerical range of  $a$  is the set

$$V_0(a) = \{f(a) : f \in \mathcal{S}_{max}(a)\},$$

where  $\mathcal{S}_{max}(a)$  is the set of all maximal states for  $a$  defined by

$$\mathcal{S}_{max}(a) := \{f \in \mathcal{S}(\mathcal{A}) : f(a^*a) = \|a\|^2\}.$$

In the same paper [7], the author established the following.

THEOREM 1.5. ([7]) *Let  $a \in \mathcal{A}$ . Then  $V_0(a)$  is a non-empty convex compact subset of  $V(a)$ . Moreover, if  $\mathcal{A} = \mathcal{B}(\mathcal{H})$  then  $V_0(a) = W_0(a)$ .*

Recall that a bounded linear operator  $A \in \mathcal{B}(\mathcal{H})$  is called quadratic if it satisfies some non-trivial quadratic equation  $(A - \alpha I)(A - \beta I) = 0$ , where  $I$  is the operator identity on  $\mathcal{H}$  and  $\alpha, \beta \in \mathbb{C}$ . We have the following.

THEOREM 1.6. ([1, 14, 18]) *Let  $A \in \mathcal{B}(\mathcal{H})$  be a quadratic operator satisfying  $(A - \alpha I)(A - \beta I) = 0$  for some scalars  $\alpha$  and  $\beta$ . Then*

(a)  *$A$  is unitarily equivalent to an operator of the form*

$$\alpha I_1 \oplus \beta I_2 \oplus \begin{bmatrix} \alpha I_3 & T \\ 0 & \beta I_3 \end{bmatrix} \text{ on } \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus (\mathcal{H}_3 \oplus \mathcal{H}_3),$$

where  $\mathcal{H}_1, \mathcal{H}_2$  and  $\mathcal{H}_3$  are complex Hilbert spaces with  $T$  being positive semi-definite on  $\mathcal{H}_3$ .

(b)

$$\|A\| = \left\| \begin{bmatrix} \alpha I_3 & T \\ 0 & \beta I_3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right\| = \frac{1}{\sqrt{2}} \sqrt{u + \sqrt{u^2 - v}},$$

where  $u = |\alpha|^2 + |\beta|^2 + \|T\|^2$  and  $v = 4|\alpha|^2|\beta|^2$ .

PROPOSITION 1.7. ([1]) *Let  $A \in \mathcal{B}(\mathcal{H})$  be a quadratic operator satisfying  $(A - \alpha I)(A - \beta I) = 0$  for some scalars  $\alpha$  and  $\beta$ . Then, the center of mass of  $A$  is*

$$c_A = \frac{\alpha + \beta}{2}.$$

THEOREM 1.8. ([10]) *Let  $A = \begin{bmatrix} \alpha & \gamma \\ 0 & \beta \end{bmatrix}$ , where  $\alpha, \beta, \gamma \in \mathbb{C}$ . Then*

$$\begin{cases} W_0(A) = \left\{ \frac{\|A\|^2(\alpha + \beta) - \alpha\beta(\overline{\alpha} + \overline{\beta})}{2\|A\|^2 - |\alpha|^2 - |\beta|^2 - |\gamma|^2} \right\}, & \text{if } \gamma \neq 0 \text{ or } |\alpha| \neq |\beta|; \\ W_0(A) = [\alpha, \beta], & \text{otherwise.} \end{cases}$$

In Section 2, we establish some results regarding hyponormal elements and normaloid elements in a complex  $C^*$ -algebra that generalize Theorem 1.2 and Theorem 1.3. We point out a gap in the proof of [7, Proposition 5.2] and give a correct proof of it. In Section 3, we provide an explicit formula for the maximal numerical range of a quadratic operator using the fact that a quadratic operator is unitarily equivalent to a direct sum of operators relatively well-known. As a corollary, we determine the maximal numerical range and the center of mass of a rank-one operator.

## 2. The algebraic maximal numerical range of a hyponormal element in a $C^*$ -algebra

We give a description of the algebraic maximal numerical range  $V_0(a)$  when  $a \in \mathcal{A}$  is hyponormal which will be a generalization of Theorem 1.2. We also give a generalization of Theorem 1.3. For this purpose, we need the following results. The first one is known as the Gelfand-Naimark theorem.

THEOREM 2.1. ([2]) *Let  $\mathcal{A}$  be a  $C^*$ -algebra with unit  $e$ . Then, there exist a complex Hilbert space  $\mathcal{H}$  and an isometric  $*$ -morphism  $T$  from  $\mathcal{A}$  onto a closed self-adjoint subalgebra  $\mathfrak{B}$  of  $\mathcal{B}(\mathcal{H})$ .*

In the sequel, we shall denote  $T(a)$  by  $T_a$  for all  $a \in \mathcal{A}$ . Therefore, we have  $\|T_a\| = \|a\|$ ,  $T_{ab} = T_a T_b$ ,  $T_e = I$  (where  $I$  is the operator identity on  $\mathcal{H}$ ) and  $T_{a^*} = (T_a)^*$  for all  $a, b \in \mathcal{A}$ . Moreover,  $a \in \mathcal{A}$  is invertible if and only if  $T_a$  is invertible. In that case,  $(T_a)^{-1} = T_{a^{-1}}$ . In particular,  $\sigma(a) = \sigma(T_a)$ . As a consequence of these properties, we have the following. Let  $a \in \mathcal{A}$ , then there is a unique scalar  $c_a$  (also called the center of mass of  $a$ ) such that

$$\|a - c_a\| = \inf_{\lambda \in \mathbb{C}} \|a - \lambda\|.$$

Moreover,  $c_a = c_{T_a}$ .

LEMMA 2.2. *Let  $a \in \mathcal{A}$ . Then the following hold*

1.  $V(a) = V(T_a)$ ;
2.  $V_0(a) = V_0(T_a)$ .

*Proof.* We give a proof of the second assertion, the proof of the first one is similar. Let  $\lambda \in V_0(a)$ . Then, there is  $f \in \mathcal{S}_{max}(a)$  such that  $f(a) = \lambda$ . Define  $g$  on  $\mathfrak{B}$  by  $g(T_x) := f(x)$  for all  $x \in \mathcal{A}$ . It is clear that  $g \in \mathcal{S}(\mathfrak{B})$ . By the Hahn-Banach theorem, we may extend  $g$  to  $\tilde{g} \in \mathcal{S}(\mathcal{B}(\mathcal{H}))$ . Moreover,

$$\tilde{g}(T_a^* T_a) = g(T_a^* T_a) = g(T_a^* a) = f(a^* a) = \|a\|^2 = \|T_a\|^2.$$

Thus,  $\tilde{g} \in \mathcal{S}_{max}(T_a)$  and since  $\tilde{g}(T_a) = g(T_a) = f(a) = \lambda$ , then  $\lambda \in V_0(T_a)$ . Consequently,  $V_0(a) \subseteq V_0(T_a)$ . A similar argument gives the other inclusion. We then obtain the desired result.  $\square$

PROPOSITION 2.3. *Let  $a \in \mathcal{A}$  be hyponormal. Then*

$$V_0(a) = co(\sigma_n(a)),$$

where  $\sigma_n(a) := \{\lambda \in \sigma(a) : |\lambda| = \|a\|\}$ .

*Proof.* Let  $a \in \mathcal{A}$  be hyponormal. It is easy to show that the operator  $T_a$  is hyponormal and  $\sigma_n(T_a) = \sigma_n(a)$ . Using Lemma 2.2 and Theorem 1.2, we get

$$V_0(a) = V_0(T_a) = co(\sigma_n(T_a)) = co(\sigma_n(a))$$

as required.  $\square$

REMARK 2.4. Let  $a \in \mathcal{A}$  and define the maximal numerical radius of  $a$  as follows

$$w_0(a) := \sup\{|\lambda| : \lambda \in V_0(a)\}.$$

From Lemma 2.2, we derive that  $w(a) = w(T_a)$  and  $w_0(a) = w_0(T_a)$ . So, since  $\|a\| = \|T_a\|$ , it follows that  $a$  is normaloid if and only if  $T_a$  is normaloid. According to Theorem 1.3, we have the following.

PROPOSITION 2.5. *Let  $a \in \mathcal{A}$ . Then  $a$  is normaloid if and only if  $w(a) = w_0(a)$ .*

In the proof of [7, Proposition 5.2], Fong used the following statement. If  $a \in \mathcal{A}$ , then  $\mathcal{S}_{max}(a) = \mathcal{S}_{max}(a^*)$ . But, this statement is not true in general. Indeed, let  $\mathcal{H} = \ell_2$  be the complex Hilbert space of square summable sequences and let  $S$  be the right shift operator on  $\mathcal{H}$  defined by  $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ . We show that  $\mathcal{S}_{max}(S) \neq \mathcal{S}_{max}(S^*)$ . It is known that  $\|S\| = 1$  and  $S^*S = I$ . Then, for any  $f \in \mathcal{S}(\mathcal{B}(\mathcal{H}))$  we have  $f(S^*S) = f(I) = 1 = \|S\|^2$ . It results that  $\mathcal{S}_{max}(S) = \mathcal{S}(\mathcal{B}(\mathcal{H}))$ . But,  $\langle SS^*(1, 0, 0, \dots), (1, 0, 0, \dots) \rangle = 0$ , so  $0 \in V(SS^*)$  and hence  $0 = g(SS^*)$  for some  $g \in \mathcal{S}(\mathcal{B}(\mathcal{H}))$ . Since  $\|S^*\| = 1$ ,  $g \notin \mathcal{S}_{max}(S^*)$ .

PROPOSITION 2.6. [7, Proposition 5.2] *Let  $a \in \mathcal{A}$ . Then*

$$V_0(a^*) = V_0(a)^*,$$

where for a subset  $\Lambda$  of  $\mathbb{C}$ ,  $\Lambda^* := \{\bar{\lambda} : \lambda \in \Lambda\}$ .

We now give a correct proof of this proposition.

*Proof.* Let  $a \in \mathcal{A}$ . According to [11, Proposition 2],  $V_0(T_a^*) = V_0(T_a)^*$  and by Lemma 2.2,  $V_0(a^*) = V_0(T_a^*) = V_0(T_a)^* = V_0(a)^*$ .  $\square$

### 3. Maximal numerical range of a quadratic operator

In this section, we calculate the maximal numerical range of a quadratic operator on a complex Hilbert space. Let  $A \in \mathcal{B}(\mathcal{H})$  be a quadratic operator satisfying the following quadratic equation  $(A - \alpha I)(A - \beta I) = 0$ , where  $\alpha, \beta \in \mathbb{C}$ . From Theorem 1.6, there exist complex Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2$  and  $\mathcal{H}_3$  such that  $A$  is unitarily equivalent to an operator of the form

$$\alpha I_1 \oplus \beta I_2 \oplus \begin{bmatrix} \alpha I_3 & T \\ 0 & \beta I_3 \end{bmatrix} \text{ on } \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus (\mathcal{H}_3 \oplus \mathcal{H}_3),$$

with  $T$  being positive semi-definite on  $\mathcal{H}_3$ . According to [11, Lemma 2],  $W_0(A) = W_0\left(\begin{bmatrix} \alpha I_3 & T \\ 0 & \beta I_3 \end{bmatrix}\right)$ . Therefore, we can assume that  $A = \begin{bmatrix} \alpha I & T \\ 0 & \beta I \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ , with  $T$  is positive. The following theorem is a generalization of Theorem 1.8.

THEOREM 3.1. *Let  $A = \begin{bmatrix} \alpha I & T \\ 0 & \beta I \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ . Then*

$$\begin{cases} W_0(A) = \left\{ \frac{\|A\|^2(\alpha + \beta) - \alpha\beta(\bar{\alpha} + \bar{\beta})}{2\|A\|^2 - |\alpha|^2 - |\beta|^2 - \|T\|^2} \right\}, & \text{if } T \neq 0 \text{ or } |\alpha| \neq |\beta|; \\ W_0(A) = [\alpha, \beta], & \text{otherwise.} \end{cases}$$

*Proof.* We show that  $W_0(A) = W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right)$  and we then conclude by Theorem 1.8. If  $T = 0$ , the result is clear since  $A$  is normal and so we apply Theorem 1.2. If  $T \neq 0$  and  $\alpha = 0$  then by Theorem 1.8,  $W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right) = \{\beta\}$ . We also have  $W_0(A) = \{\beta\}$ . Indeed, let  $\lambda \in W_0(A)$ , then there is  $x_n = y_n \oplus z_n \in \mathcal{H} \oplus \mathcal{H}$  with  $\|y_n\|^2 + \|z_n\|^2 = 1$  such that  $\lim_n \langle Ax_n, x_n \rangle = \lambda$  and  $\lim_n \|Ax_n\|^2 = \|A\|^2 = \|T\|^2 + |\beta|^2$ . Since  $\|Ax_n\|^2 = \|Tz_n\|^2 + |\beta|^2\|z_n\|^2$ , then  $\lim_n \|z_n\| = 1$  and  $\lim_n \|y_n\| = 0$ . We derive

that  $\lim_n \langle Ax_n, x_n \rangle = \lim_n (\langle Tz_n, y_n \rangle + \beta |z_n|^2) = \beta$ . Therefore, we may assume that  $T \neq 0$  and  $\alpha \neq 0$ .

We show that  $W_0(A) \subseteq W_0 \left( \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right)$ . Let  $\lambda \in W_0(A)$ , then there exists a unit vector sequence  $\{x_n\}$  in  $\mathcal{H} \oplus \mathcal{H}$  such that  $\lim_n \|Ax_n\| = \|A\|$  and  $\lim_n \langle Ax_n, x_n \rangle = \lambda$ . We decompose  $x_n$  as  $\alpha_n y_n \oplus \beta_n z_n$  where  $|\alpha_n|^2 + |\beta_n|^2 = 1$  and  $\|y_n\| = \|z_n\| = 1$ . Note that we can assume that  $\alpha \alpha_n \beta_n \geq 0$ . Therefore, we have

$$\begin{aligned} \|Ax_n\|^2 &= |\alpha|^2 |\alpha_n|^2 + 2\alpha \alpha_n \overline{\beta_n} \operatorname{Re}(\langle Tz_n, y_n \rangle) + |\beta_n|^2 \|Tz_n\|^2 + |\beta|^2 |\beta_n|^2 \\ &\leq |\alpha|^2 |\alpha_n|^2 + 2\alpha \alpha_n \overline{\beta_n} \|T\| + |\beta_n|^2 \|T\|^2 + |\beta|^2 |\beta_n|^2 \\ &= \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \right\|^2 \\ &\leq \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right\|^2 \\ &= \|A\|^2 \quad (\text{by Theorem 1.6.(b)}). \end{aligned}$$

Since  $\lim_n \|Ax_n\| = \|A\|$ , we derive that

$$\lim_n \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \right\| = \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right\|.$$

A simple computation shows that

$$\langle Ax_n, x_n \rangle = \alpha |\alpha_n|^2 + \beta_n \overline{\alpha_n} \langle Tz_n, y_n \rangle + \beta |\beta_n|^2$$

and

$$\left\langle \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}, \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \right\rangle = \alpha |\alpha_n|^2 + \beta_n \overline{\alpha_n} \|T\| + \beta |\beta_n|^2.$$

Note that the sequence  $\{\beta_n \overline{\alpha_n}\}$  is bounded, so that we may assume, by passing to a subsequence if necessary, it is convergent. If  $\lim_n \beta_n \overline{\alpha_n} = 0$ , then  $\lim_n \langle Ax_n, x_n \rangle =$

$\lim_n \left\langle \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}, \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \right\rangle$ , so  $\lambda \in W_0 \left( \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right)$ . If  $\lim_n \beta_n \overline{\alpha_n} \neq 0$ , since

$\lim_n \alpha_n \overline{\beta_n} (\operatorname{Re}(\langle Tz_n, y_n \rangle) - \|T\|) = 0$ , then  $\lim_n \operatorname{Re}(\langle Tz_n, y_n \rangle) = \|T\|$ . This implies

$\lim_n \langle Tz_n, y_n \rangle = \|T\|$  and, as above, we again have  $\lambda \in W_0 \left( \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right)$ . Conse-

quently,  $W_0(A) \subseteq W_0 \left( \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right)$ .

We now show that  $W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right) \subseteq W_0(A)$ . Let  $\lambda \in W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right)$ , then there exist  $a, b \in \mathbb{C}$  such that  $|a|^2 + |b|^2 = 1$ ,

$$\left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right\|$$

and

$$\left\langle \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right\rangle = \lambda.$$

Let  $z_n$  be unit vectors in  $\mathcal{H}$  such that  $\lim_n \|Tz_n\| = \|T\|$ . Set  $y_n := Tz_n/\|Tz_n\|$  and  $x_n := ay_n \oplus bz_n$ . We have

$$\|Ax_n\|^2 = |\alpha|^2|a|^2 + 2\operatorname{Re}(\alpha\bar{a}b)\|Tz_n\| + |b|^2\|Tz_n\|^2 + |\beta|^2|b|^2$$

and

$$\left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2 = |\alpha|^2|a|^2 + 2\operatorname{Re}(\alpha\bar{a}b)\|T\| + |b|^2\|T\|^2 + |\beta|^2|b|^2.$$

Hence

$$\lim_n \|Ax_n\| = \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \left\| \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \right\| = \|A\|.$$

On the other hand,

$$\langle Ax_n, x_n \rangle = \alpha|a|^2 + b\bar{a}\|Tz_n\| + \beta|b|^2$$

and

$$\left\langle \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right\rangle = \alpha|a|^2 + b\bar{a}\|T\| + \beta|b|^2.$$

We derive that

$$\lim_n \langle Ax_n, x_n \rangle = \left\langle \begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right\rangle = \lambda.$$

It follows that  $\lambda \in W_0(A)$ . Thus,  $W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right) \subseteq W_0(A)$ . In summary,  $W_0(A) = W_0\left(\begin{bmatrix} \alpha & \|T\| \\ 0 & \beta \end{bmatrix}\right)$ . This completes the proof.  $\square$

REMARK 3.2. An element  $a \in \mathcal{A}$  is called quadratic if there exist two scalars  $\alpha, \beta$  such that  $(a - \alpha e)(a - \beta e) = 0$ . It is clear that  $a \in \mathcal{A}$  is quadratic if and only if  $T_a$  is quadratic. Then, from Lemma 2.2, Proposition 1.7 and Theorem 3.1, we have the following.



**COROLLARY 3.3.** *Let  $a \in \mathcal{A}$  be a quadratic operator satisfying  $(a - \alpha e)(a - \beta e) = 0$  for some scalars  $\alpha$  and  $\beta$ . The algebraic maximal numerical range of  $a$  is either a point or the line segment  $[\alpha, \beta]$  connecting  $\alpha$  and  $\beta$ . Moreover, the center of mass of  $a$  is  $c_a = \frac{\alpha + \beta}{2}$ .*

As a consequence, we give a result concerning rank-one operators. Every rank-one operator is quadratic. Indeed, there exist  $f \in \mathcal{H}'$  and  $u \in \mathcal{R}(T)$  (the range of  $T$ ) such that  $T(x) = f(x)u$  for all  $x \in \mathcal{H}$ . Then  $T^2(x) - f(u)T(x) = 0$  for all  $x \in \mathcal{H}$ . That is  $T^2 - f(u)T = 0$ . Hence,  $T$  is quadratic ( $\alpha = 0$  and  $\beta = f(u)$ ). Moreover, by the Riesz representation theorem, there exists  $v \in \mathcal{H}$  such that  $f(\cdot) = \langle \cdot, v \rangle$ . Then  $T = u \otimes v$ . According to Theorem 3.1 and Proposition 1.7, we have the following result concerning the maximal numerical range and the center of mass of a rank-one operator on a complex Hilbert space.

**PROPOSITION 3.4.** *Let  $T \in \mathcal{B}(\mathcal{H})$  be a rank-one operator. Then  $W_0(T) = \{\langle u, v \rangle\}$  and  $c_T = \frac{\langle u, v \rangle}{2}$ , where  $u, v \in \mathcal{H}$  are such that  $T = u \otimes v$ .*

Note that we can obtain the previous result by observing that  $T$  is unitarily equivalent to  $\begin{bmatrix} \langle u, v \rangle & \|v\|^2 \\ 0 & 0 \end{bmatrix} \oplus 0$  and using [11, Lemma 2] and Theorem 1.8.

**REMARK 3.5.** Note that for stating Theorem 3.1 we used [11, Lemma 2] which asserts the following result. Let  $A_1 \in \mathcal{B}(\mathcal{H}_1)$  and  $A_2 \in \mathcal{B}(\mathcal{H}_2)$  where  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are complex Hilbert spaces. For  $A$  unitarily equivalent to  $A_1 \oplus A_2$ ,

$$W_0(A) = co\left(\bigcup_{\|A_j\|=\|A\|} W_0(A_j)\right).$$

This result can be generalized by induction to the finite direct sum case. But, it is not true in the infinite direct sum case in general. Indeed, let  $\{B_k\}$  for  $k = 1, 2, \dots$ , be the operators on the complex Hilbert space  $\mathcal{H} = \mathbb{C}^2$  represented by

$$B_k = \begin{bmatrix} 1 & 0 \\ 0 & -1 + \frac{1}{k} \end{bmatrix}, \quad k = 1, 2, \dots$$

It is known that  $\overline{\bigcup_k \sigma(B_k)} \subseteq \sigma(\oplus_k B_k)$ . That is,  $\overline{\bigcup_k \{-1 + \frac{1}{k}, 1\}} \subseteq \sigma(\oplus_k B_k)$ . It results that  $\{-1, 1\} \subset \sigma(\oplus_k B_k)$  and since  $\|\oplus_k B_k\| = 1$ , then  $\{-1, 1\} \subseteq \sigma_n(\oplus_k B_k)$ . From [15, Lemma 1],  $\sigma_n(\oplus_k B_k) \subseteq W_0(\oplus_k B_k)$ . We derive that  $\{-1, 1\} \subseteq W_0(\oplus_k B_k)$ . But,  $W_0(B_k) = \{1\}$ , for  $k = 1, 2, \dots$ , then  $\bigcup_k W_0(B_k) = \{1\}$ . Consequently,  $co\left(\bigcup_k W_0(B_k)\right) \subsetneq W_0(\oplus_k B_k)$ . However, we have the following.

PROPOSITION 3.6. *Let  $\{\mathcal{H}_n\}$  be a collection of complex Hilbert spaces, let  $\{T_n\}$  be a collection of hyponormal operators with  $T_n \in \mathcal{B}(\mathcal{H}_n)$ . Assume that  $\sup_n \|T_n\| < \infty$  and consider the direct sum  $T = \bigoplus_n T_n \in \mathcal{B}(\bigoplus_n \mathcal{H}_n)$ . Then*

$$W_0(T) = co\left(\overline{\bigcup_k \sigma(T_k)} \cap C_T\right),$$

where  $C_T := \{\lambda : |\lambda| = \|T\|\}$ .

*Proof.* Since  $\{T_n\}$  is a collection of hyponormal operators, then  $T$  is hyponormal. By virtue of Theorem 1.2,  $W_0(T) = co(\sigma_n(T))$ . According to [13, Proposition 2.F], we have  $\sigma(T) = \overline{\bigcup_k \sigma(T_k)}$  and the result follows.  $\square$

*Acknowledgement.* The authors would like to express their sincere gratitude to the referees for their carefully review of this paper, valuable comments and suggestions.

#### REFERENCES

- [1] A. ABU-OMAR AND P. Y. WU, *Scalar approximants of quadratic operators with applications*, Oper. Matrices, **12** (1) (2018), 253–262.
- [2] B. AUPETIT, *A primer on spectral theory*, New York (NY): Springer Verlag (1991).
- [3] A. BAGHDAD AND M. C. KAADOUD, *On the maximal numerical range of a hyponormal operator*, Oper. Matrices, **13** (4) (2019), 1163–1171.
- [4] F. F. BONSALL AND J. DUNCAN, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, London-New York: Cambridge University Press, (1971), (London mathematical society lecture note series; 2).
- [5] F. F. BONSALL AND J. DUNCAN, *Numerical ranges II*, New York-London: Cambridge University Press; (1973), (London mathematical society lecture notes series; 10).
- [6] J. T. CHAN AND K. CHAN, *An observation about normaloid operators*, Oper. Matrices, **11** (3) (2017), 885–890.
- [7] C. K. FONG, *On the essential maximal numerical range*, Acta Sci. Math., **41** (1979), 307–315.
- [8] K. E. GUSTAFSON AND D. K. M. RAO, *Numerical range: the field of values of linear operators and matrices*, Springer, New York, Inc: 1997.
- [9] P. R. HALMOS, *Hilbert space problem book*, New York: Van Nostrand (1967).
- [10] A. N. HAMED AND I. M. SPITKOVSKY, *On the maximal numerical range of some matrices*, Electron J. Linear Algebra, **34** (2018), 288–303.
- [11] G. JI, N. LIU AND Z. E. LI, *Essential numerical range and maximal numerical range of the Aluthge transform*, Linear Multilinear Algebra, **55** (4) (2007), 315–322.
- [12] D. N. KINGANGI, *On Norm of Elementary Operator: An Application of Stampfli's Maximal Numerical Range*, Pure Appl. Math. J., **7** (1) (2018), 6–10.
- [13] C. S. KUBRUSLY, *Spectral Theory of Bounded Linear Operators*, Birkhäuser (2020).
- [14] J. ROOIN, S. KARAMI AND M. G. AGHIDEH, *A new approach to numerical radius of quadratic operators*, Ann. Funct. Anal., **11** (2020), 879–896.
- [15] I. M. SPITKOVSKY, *A note on the maximal numerical range*, Oper. Matrices, **13**(3) (2019), 601–605.
- [16] J. G. STAMPFLI AND J. P. WILLIAMS, *Growth condition and the numerical range in Banach algebra*, Tohoku Math. J. **20** (1968), 417–424.

- [17] J. G. STAMPFLI, *The norm of derivation*, Pacific J. Math., **33** (1970), 737–747.  
[18] S. H. TSO AND P. Y. WU, *Matricial ranges of quadratic operators*, Rocky Mountain J. Math., **29** (1999), 1139–1152.

(Received November 4, 2020)

*E. H. Benabdi*  
Department of Mathematics, FSSM  
Cadi Ayyad University  
Marrakesh-Morocco  
e-mail: elhassan.benabdi@gmail.com

*M. Barraa*  
Department of Mathematics, FSSM  
Cadi Ayyad University  
Marrakesh-Morocco  
e-mail: barraa@uca.ac.ma

*M. K. Chraïbi*  
Department of Mathematics, FSSM  
Cadi Ayyad University  
Marrakesh-Morocco  
e-mail: chraïbik@uca.ac.ma

*A. Baghdad*  
Department of Mathematics, FSSM  
Cadi Ayyad University  
Marrakesh-Morocco  
e-mail: bagabd66@gmail.com