

EVERY REAL SYMPLECTIC MATRIX IS A PRODUCT OF COMMUTATORS OF REAL SYMPLECTIC INVOLUTIONS

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(Communicated by P. Šemrl)

Abstract. Denote by $I (I_n)$ the $(n \times n)$ identity matrix. A matrix A is symplectic if $A^T J A = J$, where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. A symplectic matrix A is a commutator of symplectic involutions if $A = XYX^{-1}Y^{-1}$, where X and Y are symplectic and $X^2 = Y^2 = I$. Let \mathbb{R} be the real number field. Denote by $\text{tr}A$ the trace of A , by $A \oplus B$ the direct sum of A and B . In this article, it is proved that every 4×4 real symplectic matrix except the matrices similar to $-I_2 \oplus B$ for $-I_2 \neq B \in \text{Sp}(2, \mathbb{R})$ and $\text{tr}B \geq -2$, can be decomposed into a product of at most two commutators of real symplectic involutions, and the exceptional real symplectic matrices are products of three commutators of real symplectic involutions. Using this result, it is shown that every real symplectic matrix of size greater than two is a product of a finite number of commutators of real symplectic involutions.

1. Introduction

An involution in a matrix group is an element A satisfying $A^2 = I$. Representations of matrices as products of involutions and commutators of involutions are interesting topics and have been studied by many scholars (see e.g. [1, 2, 4, 5, 7, 8, 9, 10, 12, 13, 14, 15]). Denote by $\text{Sp}(2n, F)$ the group consisting of all the $2n \times 2n$ symplectic matrices over a field F . A commutator of symplectic involutions is a product of two involutions (an involution and its conjugate). In [1], Awa and de la Cruz proved that every 4×4 real symplectic matrix is a product of four real symplectic involutions. In this article, we consider the problem of decomposing real symplectic matrices into products of commutators of real symplectic involutions. We wish to determine whether a 4×4 real symplectic matrix is a product of two commutators of symplectic involutions.

Denote by \oplus the matrix direct sum. Assume that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $A_{ij} \in \mathbb{R}^{m \times m}$ and $B_{ij} \in \mathbb{R}^{n \times n}$ for $i, j \in \{1, 2\}$. The *expanding sum* of A and B is defined as

$$A \boxplus B = \begin{bmatrix} A_{11} \oplus B_{11} & A_{12} \oplus B_{12} \\ A_{21} \oplus B_{21} & A_{22} \oplus B_{22} \end{bmatrix}.$$

One checks that $A \boxplus B$ is permutation similar to $A \oplus B$ and $(A \boxplus B)(C \boxplus D) = (AC) \boxplus (BD)$. The main results of this paper are the following theorems.

Mathematics subject classification (2020): 15A23, 20H25, 20H20.

Keywords and phrases: Involutions, commutators of involutions, symplectic matrices.

THEOREM 1.1. *Every 4×4 real symplectic matrix except the matrices similar to $-I_2 \boxplus B$ for $-I_2 \neq B \in \text{Sp}(2, \mathbb{R})$ and $\text{tr}B \geq -2$, is a product of at most two commutators of real symplectic involutions. The exceptional 4×4 real symplectic matrices are products of three commutators of real symplectic involutions.*

THEOREM 1.2. *Every $2n \times 2n$ real symplectic matrix is a product of a finite number of commutators of real symplectic involutions.*

The proofs of these two theorems will be given at the end of section 3. We will give some preliminaries in section 2 first.

2. Preliminaries

To prove our main theorems we need the following remarks and lemmas.

REMARK 2.1. Assume that $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where $A_i \in \mathbb{R}^{n \times n}$ for $i \in \{1, 2, 3, 4\}$. Then A is symplectic if and only if

$$A_1A_2^T, A_3A_4^T \text{ are symmetric, and } A_1A_4^T - A_2A_3^T = I_n.$$

Then the inverse matrix of A is

$$A^{-1} = \begin{bmatrix} A_4^T & -A_2^T \\ -A_3^T & A_1^T \end{bmatrix}.$$

REMARK 2.2. $A \boxplus B$ is symplectic if and only if A and B are symplectic.

REMARK 2.3. Denote by $\mathcal{C}\mathcal{S}\mathcal{I}(m)$ the set of all products of m commutators of real symplectic involutions.

- (a) $\mathcal{C}\mathcal{S}\mathcal{I}(1) \subset \mathcal{C}\mathcal{S}\mathcal{I}(2) \subset \mathcal{C}\mathcal{S}\mathcal{I}(3) \subset \dots$
- (b) Each element of the sets $\mathcal{C}\mathcal{S}\mathcal{I}(m)$ is closed under expanding summation, i.e., if $A \in \mathcal{C}\mathcal{S}\mathcal{I}(m)$, $B \in \mathcal{C}\mathcal{S}\mathcal{I}(m)$, then $A \boxplus B \in \mathcal{C}\mathcal{S}\mathcal{I}(m)$.
- (c) Each element of the sets $\mathcal{C}\mathcal{S}\mathcal{I}(m)$ is invariant under real symplectic similarity, i.e., if $A \in \mathcal{C}\mathcal{S}\mathcal{I}(m)$ and P is a real symplectic matrix of the same size as A , then $P^{-1}AP \in \mathcal{C}\mathcal{S}\mathcal{I}(m)$.
- (d) $A \in \mathcal{C}\mathcal{S}\mathcal{I}(m)$ if and only if $A^{-1} \in \mathcal{C}\mathcal{S}\mathcal{I}(m)$, if and only if $A^T \in \mathcal{C}\mathcal{S}\mathcal{I}(m)$.

The following lemma is a direct consequence of the canonical form of [6] Theorem 1, see also [1, 11].

LEMMA 2.4. *Each 4×4 real symplectic matrix is symplectically similar to one of the following matrices:*

$$(1) \mathcal{P}_1 = \begin{bmatrix} \lambda^{-1} & -\lambda^{-2} & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{bmatrix}, \text{ where } \lambda \in \mathbb{R} \setminus \{0\}.$$

$$(2) \mathcal{P}_2(\alpha, \mu) = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & \alpha\mu & \alpha \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 1 & \mu \end{bmatrix}, \text{ where } \alpha \in \{-1, 0, 1\}, \mu \in \{-1, 1\}.$$

$$(3) \mathcal{P}_3 = \begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a/(a^2+b^2) & b/(a^2+b^2) \\ 0 & 0 & -b/(a^2+b^2) & a/(a^2+b^2) \end{bmatrix}, \text{ where } a, b \in \mathbb{R} \text{ such that } a^2 + b^2 \neq 1.$$

$$(4) \mathcal{P}_4 = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & \alpha & -\alpha \cot \theta \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix}, \text{ where } \theta \in (0, 2\pi) \setminus \{\pi\} \text{ and } \alpha \in \{-1, 0, 1\}.$$

(5) $\mathcal{P}_5 = A \boxplus B$ for some 2×2 real symplectic matrices A and B .

The following lemma is a direct consequence of [1] Theorem 2.

LEMMA 2.5. *Let A be a real symplectic matrix. The following are equivalent.*

- (1) A is a commutator of real symplectic involutions.
- (2) there exists a real symplectic matrix B such that $B^2 = A$ and B is a product of two real symplectic involution.
- (3) there exists a real symplectic matrix B such that $B^2 = A$ and B is similar to B^{-1} via a real symplectic involution.

LEMMA 2.6. ([2], Theorem 8) *A symplectic matrix A is a product of two symplectic involutions if and only if each number $\eta_k(A, \lambda)$ of $k \times k$ Jordan blocks of A corresponding to an eigenvalue λ is even.*

LEMMA 2.7. ([8], Lemma 2.5) *A symplectic matrix A is a commutator of complex symplectic involutions if and only if each number $\eta_k(A, \lambda)$ ($\lambda \neq -1$) of $k \times k$ Jordan blocks of A corresponding to an eigenvalue λ is even, and each number $\eta_k(A, -1)$ is divisible by 4.*

For a matrix $A \in \mathbb{R}^{n \times n}$, by $\sigma(A)$ we denote the spectrum of A .

LEMMA 2.8. ([1], Lemma 6) *Let $A, B \in \mathbb{R}^{n \times n}$ and suppose that $M = \begin{bmatrix} A & B \\ 0 & A^{-T} \end{bmatrix}$ is symplectic. If $\sigma(A) \cap \sigma(A^{-1}) = \emptyset$, then M is symplectically similar to $A \oplus A^{-T}$.*

LEMMA 2.9. ([1], Lemma 5) *Let A be a 2×2 nonscalar real symplectic matrix.*

- (1) *If A is not a diagonal matrix, then A is real symplectically similar to $L_{A, \mu} = \begin{bmatrix} \text{tr} A & \mu \\ -\mu & 0 \end{bmatrix}$, for some $\mu \in \{1, -1\}$.*

(2) If $|\text{tr}A| > 2$, then A is real symplectically similar to both $L_{A,1}$ and $L_{A,-1}$. In particular, A is real symplectically similar to $\text{diag}(\lambda, \lambda^{-1})$.

LEMMA 2.10. Let A be a 2×2 nonscalar real symplectic matrix. If $\text{tr}A = -2$, then A is real symplectically similar to $\begin{bmatrix} -1 & \mu \\ 0 & -1 \end{bmatrix}$, for some $\mu \in \{1, -1\}$.

Proof. Since A is nonscalar and $\text{tr}A = -2$, the matrix A must be not a diagonal matrix. By Lemma 2.9(1), A is real symplectically similar to $L_{A,\mu} = \begin{bmatrix} -2 & \mu \\ -\mu & 0 \end{bmatrix}$, for some $\mu \in \{1, -1\}$. Let $P = \begin{bmatrix} \mu & -1 \\ 1 & 0 \end{bmatrix}$, which is symplectic. One checks that $P^{-1}L_{A,\mu}P = \begin{bmatrix} -1 & \mu \\ 0 & -1 \end{bmatrix}$, as desired. \square

LEMMA 2.11. Let $A = [a_{ij}]_{2 \times 2} \in \text{Sp}(2, \mathbb{R})$. If $|\text{tr}A| < 2$, then one of the following statements holds.

(1) A is real symplectically similar to $\begin{bmatrix} \text{tr}A & 1 \\ -1 & 0 \end{bmatrix}$ for $a_{12} > 0$.

(2) A is real symplectically similar to $\begin{bmatrix} \text{tr}A & -1 \\ 1 & 0 \end{bmatrix}$ for $a_{12} < 0$.

Proof. Since $|\text{tr}A| < 2$ and the determinant of A is 1, a_{12} must be a nonzero real number. Let

$$P = \begin{bmatrix} a_{11}\sqrt{|a_{12}|}/a_{12}^{-1} & \sqrt{|a_{12}|} \\ -\sqrt{|a_{12}|} & 0 \end{bmatrix},$$

which is symplectic. If $a_{12} > 0$, then $PAP^{-1} = \begin{bmatrix} \text{tr}A & 1 \\ -1 & 0 \end{bmatrix}$. While if $a_{12} < 0$, then

$$PAP^{-1} = \begin{bmatrix} \text{tr}A & -1 \\ 1 & 0 \end{bmatrix}.$$

Now we prove that $\begin{bmatrix} \text{tr}A & 1 \\ -1 & 0 \end{bmatrix}$ is not real symplectically similar to $\begin{bmatrix} \text{tr}A & -1 \\ 1 & 0 \end{bmatrix}$ for $|\text{tr}A| < 2$. If the two matrices are real symplectically similar, then there exists a real symplectic matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \text{ satisfying } \begin{bmatrix} \text{tr}A & 1 \\ -1 & 0 \end{bmatrix} B = B \begin{bmatrix} \text{tr}A & -1 \\ 1 & 0 \end{bmatrix}.$$

Then we get $b_{12} = b_{21}$ and $-b_{11} - b_{22} = b_{12}\text{tr}A$. Thus $(b_{12}\text{tr}A)^2 = (-b_{11} - b_{22})^2 \geq 4b_{11}b_{22}$. Since the determinant of B is 1 and $b_{12} = b_{21}$, we obtain $b_{11}b_{22} = 1 + b_{12}^2$. So $(b_{12}\text{tr}A)^2 \geq 4b_{11}b_{22} = 4 + 4b_{12}^2$. Recall that $|\text{tr}A| < 2$ we get $4 + 4b_{12}^2 \leq$

$(b_{12}\text{tr}A)^2 \leq 4b_{12}^2$, a contradiction. Thus $\begin{bmatrix} \text{tr}A & 1 \\ -1 & 0 \end{bmatrix}$ is not real symplectically similar to $\begin{bmatrix} \text{tr}A & -1 \\ 1 & 0 \end{bmatrix}$. \square

3. Proof of the main result

LEMMA 3.1. Let $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \in \text{Sp}(2, \mathbb{R})$ for $\theta \in (0, 2\pi) \setminus \{\pi\}$. Then $A \boxplus A$ is not a commutator of real symplectic involutions.

Proof. Since a commutator of real symplectic involutions is a product of two real symplectic involutions, it suffices to prove that $A \boxplus A$ is not a product of two real symplectic involutions. By [1] Theorem 2, there must exist a real symplectic matrix P such that $P^{-1}(A \boxplus A)P = (A \boxplus A)^{-1}$. Write

$$P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$$

where each P_i is a 2×2 real matrix for $i \in \{1, 2, 3, 4\}$. Then from $(A \boxplus A)P = P(A \boxplus A)^{-1}$ we get $P_2 = P_3$ and $P_1 = -P_4$. From Remark 2.1 we know that $P_1 P_4^T - P_2 P_3^T = I_n = -P_4 P_4^T - P_3 P_3^T$. Since P_3 and P_4 are real, we get a contradiction. Thus $A \boxplus A$ is neither a product of two real symplectic involutions nor a commutator of real symplectic involutions. \square

LEMMA 3.2. Let $0 < \lambda \in \mathbb{R}$, $\theta \in [0, 2\pi)$, $a \in \mathbb{R} \setminus \{0\}$, $b, c \in \mathbb{R}$. The following symplectic matrices are commutators of real symplectic involutions.

$$(1) H_1 = \begin{bmatrix} \lambda^{-1} & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

$$(2) H_2 = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix}.$$

$$(3) H_3 = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & \cos \theta & 0 & -\sin \theta \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & \sin \theta & 0 & \cos \theta \end{bmatrix}.$$

$$(4) H_4 = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & -a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } H_4^T.$$

$$(5) H_5 = \begin{bmatrix} 1 & a & c & b \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 1 \end{bmatrix} \text{ and } H_5^T.$$

Proof. Choose

$$K_1 = \begin{bmatrix} \sqrt{\lambda^{-1}} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda^{-1}} & 0 & 0 \\ 0 & 0 & \sqrt{\lambda} & 0 \\ 0 & 0 & 0 & \sqrt{\lambda} \end{bmatrix}.$$

One checks that K_1 is real symplectically similar to its inverse by the real symplectic involution

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since $H_1 = K_1^2$, by Lemma 2.5, H_1 is a commutator of real symplectic involutions.

Choose

$$K_2 = \begin{bmatrix} \cos(\theta/2) & \sin(\theta/2) & 0 & 0 \\ -\sin(\theta/2) & \cos(\theta/2) & 0 & 0 \\ 0 & 0 & \cos(\theta/2) & \sin(\theta/2) \\ 0 & 0 & -\sin(\theta/2) & \cos(\theta/2) \end{bmatrix},$$

$$K_3 = \begin{bmatrix} \cos(\theta/2) & 0 & \sin(\theta/2) & 0 \\ 0 & \cos(\theta/2) & 0 & -\sin(\theta/2) \\ -\sin(\theta/2) & 0 & \cos(\theta/2) & 0 \\ 0 & \sin(\theta/2) & 0 & \cos(\theta/2) \end{bmatrix}.$$

One checks that both K_2 and K_3 are real symplectically similar to their inverse by the same real symplectic involution

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Since $H_2 = K_2^2$ and $H_3 = K_3^2$, by Lemma 2.5, H_2 and H_3 are commutators of real symplectic involutions.

Next let

$$K_4 = \begin{bmatrix} 1 & 0 & a/2 & 0 \\ 0 & 1 & 0 & -a/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

One checks that K_4 is symplectic and $K_4^2 = H_4$, and that K_4 is real symplectically similar to its inverse by the real symplectic involution P . Thus by Lemma 2.5, H_4 is a commutator of real symplectic involutions and so is H_4^T .

Let

$$K_5 = \begin{bmatrix} 1 & a/2 & c/2 & b/2 \\ 0 & 1 & b/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a/2 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 & 0 & a^{-1}c \\ 0 & -1 & -a^{-1}c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

One checks that both K_5 and Q are symplectic and $K_5^2 = H_3$, $Q^2 = I$, $Q^{-1}K_5Q = K_5^{-1}$. Thus by Lemma 2.5, H_5 is a commutator of real symplectic involutions and so is H_5^T . \square

LEMMA 3.3. $-I_4$ is a commutator of real symplectic involutions.

Proof. Since $-I_4 = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)^2$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is similar to its inverse by the real symplectic involution $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we get the conclusion by Lemma 2.5. \square

LEMMA 3.4. \mathcal{P}_1 is a product of at most two commutators of real symplectic involutions.

Proof. If $\lambda = 1$, by Lemma 3.2(5) we know that \mathcal{P}_1 is a commutator of real symplectic involutions. If $\lambda = -1$, by Lemma 3.2(5) we know that $-\mathcal{P}_1$ is a commutator of real symplectic involutions. By Lemma 3.3, $-I_4$ is a commutator of real symplectic involutions. Thus $\mathcal{P}_1 = (-\mathcal{P}_1)(-I_4)$ is a product of at most two commutators of real symplectic involutions.

Next we assume that $\lambda \neq \pm 1$. We claim that \mathcal{P}_1 is real symplectically similar to

$$K = \begin{bmatrix} 1 & 1 & -1 & 1 \\ t & 1+t & 1-t & t \\ 0 & 0 & 1+t & -t \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

where $t = \lambda + \lambda^{-1} - 2$. Note that

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

then \mathcal{P}_1 is a product of two commutators of real symplectic involutions by Lemma 3.2(5).

Now we prove that \mathcal{P}_1 is real symplectically similar to K . Let

$$P_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \lambda^{-1} & \\ & & & -1 \end{bmatrix} \oplus \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \lambda^{-1} & \\ & & & -1 \end{bmatrix}^{-T},$$

$$P_2 = \begin{bmatrix} 1 & -2\lambda^2(1-\lambda)^{-2}(1+\lambda)^{-3} \\ & 1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & -2\lambda^3(1-\lambda)^{-2}(1+\lambda)^{-3} \\ & 1 \end{bmatrix}.$$

Then we can calculate that

$$(P_1P_2)^{-1}K(P_1P_2) = \begin{bmatrix} \lambda & 0 & 0 & -\lambda^2(\lambda+1)^{-2} \\ 0 & \lambda^{-1} & -(\lambda+1)^{-2} & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Let

$$P_3 = \begin{bmatrix} \lambda(1+\lambda)^{-1} & 0 & 0 & 0 \\ 0 & \lambda(1+\lambda)^{-1} & 0 & 0 \\ 0 & 0 & \lambda^{-1}(1+\lambda) & 0 \\ 0 & 0 & 0 & \lambda^{-1}(1+\lambda) \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

One checks that $(P_1P_2P_3P_4)^{-1}K(P_1P_2P_3P_4) = \mathcal{P}_1$. Since P_1, P_2, P_3, P_4 are all real symplectic matrices, we finish the proof of the claim. \square

LEMMA 3.5. $\mathcal{P}_2(\alpha, \mu)$ is a product of two commutators of real symplectic involutions.

Proof. Since

$$\mathcal{P}_2(\alpha, 1) = \begin{bmatrix} 1 & -1 & \alpha & \alpha \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\alpha & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

by Lemma 3.2(4)(5) it is a product of two commutators of real symplectic involutions.

Let

$$K = \begin{bmatrix} 1 & -2 & -\alpha & \alpha \\ 2 & -3 & -\alpha & 2\alpha \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} -2 & 0 & 0 & \alpha/4 \\ -2 & -1 & \alpha/8 & -3\alpha/4 \\ 0 & 0 & -1/2 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

One checks that both K and P are symplectic and $\mathcal{P}_2(\alpha, -1) = P^{-1}KP$. Since $\mathcal{P}_2(\alpha, -1)$ is real symplectically similar to

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -\alpha & \alpha \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix},$$

by Lemma 3.2(5) it is a product of two commutators of real symplectic involutions. \square

LEMMA 3.6. \mathcal{P}_3 is a product of two commutators of real symplectic involutions.

Proof. Let $\lambda = \sqrt{a^2 + b^2}^{-1}$. Then there exist an angle θ such that $\cos \theta = a\lambda$ and $\sin \theta = b\lambda$. Hence,

$$\mathcal{P}_3 = \begin{bmatrix} \lambda^{-1} & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix}$$

is a product of two commutators of real symplectic involutions by Lemma 3.2(1)(2). \square

LEMMA 3.7. \mathcal{P}_4 is a product of two commutators of real symplectic involutions.

Proof. Let

$$K_1 = \begin{bmatrix} 1 & 0 & -2\alpha \csc \theta & 0 \\ 0 & 1 & 0 & \alpha \csc \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} \cos \theta & \sin \theta & 2\alpha \cot \theta & 2\alpha \\ -\sin \theta & \cos \theta & 2\alpha & -2\alpha \cot \theta \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix}.$$

Then $\mathcal{P}_4 = K_1 K_2$. Let

$$P_1 = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 2\alpha \csc \theta & -\alpha \cot \theta \csc \theta \\ 0 & 1 & -\alpha \cot \theta \csc \theta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

One checks that P_1 and P_2 are symplectic and

$$P_1^{-1} K_1 P_1 = \begin{bmatrix} 1 & 0 & -\alpha \csc \theta & 0 \\ 0 & 1 & 0 & \alpha \csc \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad P_2^{-1} K_2 P_2 = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix},$$

which are commutators of real symplectic involutions by Lemma 3.2(2)(4). Thus we get the conclusion. \square

LEMMA 3.8. For each nonscalar matrices $A, B \in \text{Sp}(2, \mathbb{R})$, $A \boxplus B$ is a product of at most two commutators of real symplectic involutions.

Proof. By Lemma 2.9(1), A is real symplectically similar to $\begin{bmatrix} \text{tr} A & \mu_1 \\ -\mu_1 & 0 \end{bmatrix}$ and B is real symplectically similar to $\begin{bmatrix} \text{tr} B & \mu_2 \\ -\mu_2 & 0 \end{bmatrix}$ for $\mu_1, \mu_2 \in \{1, -1\}$. Thus $A \boxplus B$ is real symplectically similar to $\begin{bmatrix} \text{tr} A & \mu_1 \\ -\mu_1 & 0 \end{bmatrix} \boxplus \begin{bmatrix} \text{tr} B & \mu_2 \\ -\mu_2 & 0 \end{bmatrix}$. Let $1 < \lambda \in \mathbb{R}$.

$$\begin{aligned} \begin{bmatrix} \text{tr} A & \mu_1 \\ -\mu_1 & 0 \end{bmatrix} \boxplus \begin{bmatrix} \text{tr} B & \mu_2 \\ -\mu_2 & 0 \end{bmatrix} &= \left(\begin{bmatrix} \lambda & \mu_1(\lambda + \lambda^{-1} - \lambda^{-1} \text{tr} A) \\ 0 & \lambda^{-1} \end{bmatrix} \boxplus \begin{bmatrix} \lambda & \mu_2(\lambda + \lambda^{-1} - \lambda^{-1} \text{tr} B) \\ 0 & \lambda^{-1} \end{bmatrix} \right) \\ &\quad \cdot \left(\begin{bmatrix} \lambda + \lambda^{-1} & \mu_1 \lambda^{-1} \\ -\mu_1 \lambda & 0 \end{bmatrix} \boxplus \begin{bmatrix} \lambda + \lambda^{-1} & \mu_2 \lambda^{-1} \\ -\mu_2 \lambda & 0 \end{bmatrix} \right). \end{aligned}$$

By Lemma 2.9(2), the two symplectic matrices on the right side of the above equation are all real symplectically similar to $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \boxplus \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$. Then by Lemma 3.2(1), we get the conclusion of this lemma. \square

LEMMA 3.9. *If $-I_2 \neq B \in \text{Sp}(2, \mathbb{R})$, then $I_2 \boxplus B$ is a product of at most two commutators of real symplectic involutions.*

Proof. By Lemma 2.9(1), B is real symplectically similar to $\begin{bmatrix} \text{tr}B & \mu \\ -\mu & 0 \end{bmatrix}$ for some $\mu \in \{1, -1\}$. Let $1 < \lambda \in \mathbb{R}$. Since

$$I_2 \boxplus \begin{bmatrix} \text{tr}B & \mu \\ -\mu & 0 \end{bmatrix} = \left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \boxplus \begin{bmatrix} \lambda & \mu(\lambda + \lambda^{-1} - \lambda^{-1}\text{tr}B) \\ 0 & \lambda^{-1} \end{bmatrix} \right) \cdot \left(\begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} \boxplus \begin{bmatrix} \lambda + \lambda^{-1} & \mu\lambda^{-1} \\ -\mu\lambda & 0 \end{bmatrix} \right),$$

and the two symplectic matrices on the right side of the above equation are all real symplectically similar to $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \boxplus \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ by Lemma 2.9(2). Then by Lemma 3.2(1), we get the conclusion of this lemma. \square

LEMMA 3.10. *Let $A = (-I_2) \boxplus B$, where $-I_2 \neq B \in \text{Sp}(2, \mathbb{R})$. If $\text{tr}B \geq -2$, then A is a product of three commutators of real symplectic involutions and no fewer.*

Proof. By Lemma 2.7, A can not be a commutator of real symplectic involutions. Suppose that $A = A_1A_2$ where A_1 and A_2 are commutators of real symplectic involutions. Since each number of Jordan blocks of A_k ($k \in \{1, 2\}$) is even, the degree of the minimal polynomial of A_k is at most 2 by Lemma 2.7. So there exist a monic polynomial $p_k(x)$ of degree two such that $p_k(A_k) = O$ and the characteristic polynomial of A_k equals $p_k^2(x)$. Since the determinant of symplectic matrix is 1 and each number of Jordan blocks of A_k ($k \in \{1, 2\}$) is even, one can set $p_k(x) = x^2 - a_kx + 1$ for some $a_k \in \mathbb{R}$. One obtains $A_k + A_k^{-1} = a_kI$ since $p_k(A_k) = O$. Write

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \text{Sp}(2, \mathbb{R}), \quad A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \in \text{Sp}(4, \mathbb{R}).$$

Then from Remark 2.1 one get

$$A^{-1} = (-I_2) \boxplus \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix}, \quad A_1^{-1} = \begin{bmatrix} a_{33} & a_{43} & -a_{13} & -a_{23} \\ a_{34} & a_{44} & -a_{14} & -a_{24} \\ -a_{31} & -a_{41} & a_{11} & a_{21} \\ -a_{32} & -a_{42} & a_{12} & a_{22} \end{bmatrix}$$

Compare each entry on both sides of the two matrix equation $A_1 + A_1^{-1} = a_1 I_4$ one obtains

$$\begin{cases} a_{11} + a_{33} = a_{22} + a_{44} = a_1, \\ a_{12} = -a_{43}, \\ a_{14} = a_{23}, \\ a_{21} = -a_{34}, \\ a_{32} = a_{41}. \end{cases} \tag{3.1}$$

Since $A_2 = A_1^{-1}A$, compare each entry on both sides of the two matrix equation $A_2 + A_2^{-1} = a_2 I_4$ one obtains

$$\begin{cases} a_{11} + a_{33} = -a_{22}b_{22} + a_{24}b_{21} + a_{42}b_{12} - a_{44}b_{11} = -a_2, \\ a_{12} = -a_{23}b_{21} + a_{43}b_{11}, \\ a_{14} = -a_{23}b_{22} + a_{43}b_{12}, \\ a_{32} = a_{21}b_{21} - a_{41}b_{11}, \\ a_{34} = a_{21}b_{22} - a_{41}b_{12}. \end{cases} \tag{3.2}$$

Combining the last four lines of equation systems (3.1) and (3.2), one gets

$$\begin{cases} a_{23}b_{21} = a_{43}(1 + b_{11}), \\ a_{23}(1 + b_{22}) = a_{43}b_{12}, \\ a_{21}b_{21} = a_{41}(1 + b_{11}), \\ a_{21}(1 + b_{22}) = a_{41}b_{12} \end{cases}$$

which is equivalent to

$$\begin{bmatrix} 1 + b_{22} & -b_{12} \\ -b_{21} & 1 + b_{11} \end{bmatrix} \begin{bmatrix} a_{23} \\ a_{43} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 + b_{22} & -b_{12} \\ -b_{21} & 1 + b_{11} \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{41} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If $\text{tr}B > -2$, one checks

$$\begin{vmatrix} 1 + b_{22} & -b_{12} \\ -b_{21} & 1 + b_{11} \end{vmatrix} \neq 0$$

since $B \in \text{Sp}(2, \mathbb{R})$. Thus

$$a_{12} = a_{14} = a_{21} = a_{23} = a_{32} = a_{34} = a_{41} = a_{43} = 0. \tag{3.3}$$

If $\text{tr}B = -2$, then by Lemma 2.10, B is real symplectically similar to $J_{B,\mu} := \begin{bmatrix} -1 & \mu \\ 0 & -1 \end{bmatrix}$ for some $\mu \in \{1, -1\}$ since $-I_2 \neq B \in \text{Sp}(2, \mathbb{R})$. By Remark 2.3(c), it suffices to prove that $(-I_2) \boxplus J_{B,\mu}$ is a product of three commutators of real symplectic involutions and no fewer. Replacing B by $J_{B,\mu}$, we obtain the following equation

system from (3.2).

$$\begin{cases} a_{11} + a_{33} = a_{22} + \mu a_{42} + a_{44} = -a_2, \\ a_{12} = -a_{43}, \\ a_{14} = a_{23} + \mu a_{43}, \\ a_{32} = a_{41}, \\ a_{34} = -a_{21} - \mu a_{41}. \end{cases} \tag{3.4}$$

Combining equation systems (3.1) and (3.4), we get $a_{12} = a_{32} = a_{41} = a_{42} = a_{43} = 0$ and $a_{11} + a_{33} = a_{22} + a_{44} = a_1 = -a_2 = \text{tr}B/2 = -1$. Since $A_1 \in \text{Sp}(4, \mathbb{R})$, by Remark 2.1,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}^T - \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.5}$$

Comparing the (4, 4)-entry of both sides of equation (3.5) we obtain $a_{22}a_{44} = 1$ (recall that $a_{12} = a_{32} = a_{41} = a_{42} = a_{43} = 0$). Combining $a_{22} + a_{44} = -1$ and $a_{22}a_{44} = 1$, we obtain a_{22} and a_{44} are not real numbers. Thus in this case, there does not exist real symplectic matrix A_1 such that $A = A_1A_2$ and A_1 and A_2 are commutators of real symplectic involutions.

Next we prove this lemma for $\text{tr}B > -2$. In this case, we can write A_1 as

$$A_1 = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} \boxplus \begin{bmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{bmatrix}$$

since equation (3.3) holds. Now we write $A_1 = A_0 \boxplus B_1$, where

$$A_0 = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}, \quad B_1 = \begin{bmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{bmatrix}.$$

Then $A_2 = -A_0^{-1} \boxplus B_2$ for $B_2 = B_1^{-1}B$. Since A_1 and A_2 are symplectic, one easily get that $A_0, B_1, -A_0^{-1}, B_2$ are symplectic by Remark 2.2.

If one of A_1 and A_2 has real eigenvalues, then by Lemma 2.7, either A_1 or A_2 has eigenvalues $\lambda, \lambda, \lambda^{-1}, \lambda^{-1}$ for some real number $\lambda < 0$. Denote it by A_3 . Recall that A_1 and A_2 are commutators of real symplectic involutions. By Lemma 2.5, there must exist a real symplectic matrix C such that $C^2 = A_3$ and C is a product of two real symplectic involutions. Since A_3 has negative eigenvalues $\lambda, \lambda, \lambda^{-1}, \lambda^{-1}$, C must have eigenvalues $\sqrt{-\lambda}i, -\sqrt{-\lambda}i, \sqrt{-\lambda^{-1}}i, -\sqrt{-\lambda^{-1}}i$. By Lemma 2.6, we get that $\lambda = -1$. In this case, one of A_1 and A_2 is $-I_4$ and the other is $I \boxplus (-B)$ by Lemma 2.7. Since $B \neq -I_2$, $I \boxplus (-B)$ can not be a commutator of real symplectic involutions by Lemma 2.7, a contradiction. Thus both A_1 and A_2 have no real eigenvalues. So $p_1(A_1) = p_1(A_0) = p_1(B_1)$, $p_2(A_2) = p_2(-A_0^{-1}) = p_2(B_2)$, $\text{tr}A_0 = \text{tr}B_1 = a_1 = -\text{tr}(-A_0^{-1}) = -\text{tr}B_2 = -a_2$ and $|a_1| < 2$, $|a_2| < 2$. Then by Lemma 2.11, A_0 is real symplectically similar to either $\begin{bmatrix} a_1 & 1 \\ -1 & 0 \end{bmatrix}$ or $\begin{bmatrix} a_1 & -1 \\ 1 & 0 \end{bmatrix}$, and so is B_1 . We suppose that A_0 is real symplectically similar to $\begin{bmatrix} a_1 & 1 \\ -1 & 0 \end{bmatrix}$. The other case can be proven similarly. Then

$-A_0^{-1}$ is real symplectically similar to $\begin{bmatrix} -a_1 & 1 \\ -1 & 0 \end{bmatrix}$ by Lemma 2.11. In this case, B_1 must be real symplectically similar to $\begin{bmatrix} a_1 & -1 \\ 1 & 0 \end{bmatrix}$. Otherwise A_1 is real symplectically similar to some symplectic matrix described in Lemma 3.1 with $\theta = \arccos(a_1/2)$, which is not a commutator of real symplectic involutions. For the same reason, B_2 must be real symplectically similar to $\begin{bmatrix} -a_1 & -1 \\ 1 & 0 \end{bmatrix}$. So we can find a real symplectic matrix $P = P_1 \boxplus P_2$ such that $P^{-1}A_1P = \begin{bmatrix} a_1 & 1 \\ -1 & 0 \end{bmatrix} \boxplus \begin{bmatrix} a_1 & -1 \\ 1 & 0 \end{bmatrix}$. Since $A = (-I_2) \boxplus B = A_1A_2$, $P^{-1}A_2P = \begin{bmatrix} 0 & 1 \\ -1 & -a_1 \end{bmatrix} \boxplus \hat{B}_2$ where $\hat{B}_2 = P_2^{-1}B_2P_2$. Then $\hat{B} := P_2^{-1}BP_2 = \begin{bmatrix} a_1 & -1 \\ 1 & 0 \end{bmatrix} \hat{B}_2$ and \hat{B}_2 is real symplectically similar to $\begin{bmatrix} -a_1 & -1 \\ 1 & 0 \end{bmatrix}$. Suppose the first row of \hat{B}_2 is $[x, y]$ for $x, y \in \mathbb{R}$. By Lemma 2.11, $y < 0$. We can calculate the second row of \hat{B}_2 since $\text{tr}\hat{B}_2 = -a_1$ and $\det\hat{B}_2 = 1$. So

$$\hat{B}_2 = \begin{bmatrix} x & y \\ -y^{-1}(1 + a_1x + x^2) & -a_1 - x \end{bmatrix}.$$

Thus

$$\begin{aligned} \text{tr}B &= \text{tr}\hat{B} = \text{tr} \left(\begin{bmatrix} a_1 & -1 \\ 1 & 0 \end{bmatrix} \hat{B}_2 \right) \\ &= \text{tr} \left(\begin{bmatrix} a_1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ -y^{-1}(1 + a_1x + x^2) & -a_1 - x \end{bmatrix} \right) \\ &= a_1x + y^{-1}(1 + a_1x + x^2) + y \\ &= -2 + y^{-1} \left(\left(x + \frac{a_1(y+1)}{2} \right)^2 + (y+1)^2 - \frac{a_1^2(y+1)^2}{4} \right). \end{aligned}$$

Note that $y^{-1} < 0$ and $a_1^2 < 4$ we get that $\text{tr}B \leq -2$, a contradiction. Thus A can not be expressed as a product of two commutators of real symplectic involutions.

Now we prove that A is a product of three commutators of real symplectic involutions. If $B \neq I_2$, $-A$ is a product of at most two commutators of real symplectic involutions by Lemma 3.9. Then $A = (-A)(-I_4)$ is a product of three commutators of real symplectic involutions since $-I_4$ is a commutator of real symplectic involutions by Lemma 3.3. If $B = I_2$, then

$$A = \text{diag}(-1, 1, -1, 1) = \left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right)$$

and $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is a commutator of real symplectic involutions from Lemma 3.2(4), it suffices to prove that $\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is a product of two commutators of

real symplectic involutions. Let $1 < \lambda \in \mathbb{R}$. Then

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} &= \begin{bmatrix} \lambda & -\lambda^{-1} \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} \lambda^{-1} & 0 \\ 4\lambda & \lambda \end{bmatrix}, \\ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} \lambda & -\lambda^{-1} \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix}. \end{aligned}$$

One can easily check that $\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is real symplectically similar to the product of $\begin{bmatrix} \lambda & -\lambda^{-1} \\ 0 & \lambda^{-1} \end{bmatrix} \boxplus \begin{bmatrix} \lambda & -\lambda^{-1} \\ 0 & \lambda^{-1} \end{bmatrix}$ and $\begin{bmatrix} \lambda^{-1} & 0 \\ 4\lambda & \lambda \end{bmatrix} \boxplus \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix}$. By Lemma 2.8, these two matrices are all real symplectically similar to $\text{diag}(\lambda, \lambda, \lambda^{-1}, \lambda^{-1})$, which are commutators of real symplectic involutions by Lemma 3.2(1). Thus $\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is a product of two commutators of real symplectic involutions, as desired. \square

LEMMA 3.11. *Let $A = (-I_2) \boxplus B$. If $\text{tr}B < -2$, then A is a product of two commutators of real symplectic involutions.*

Proof. Let $x = \sqrt{-2 - \text{tr}B}$. By Lemma 2.9, all the real symplectic 2×2 matrices whose traces are equal to $\text{tr}B$, are real symplectically similar since the similar relation is transitive. So B is real symplectically similar to

$$\begin{bmatrix} -1 - x^2 & x \\ x & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & -1 \\ 1 + x^2 & -x \end{bmatrix},$$

whose trace is equal to $\text{tr}B$. Thus $A = (-I_2) \boxplus B$ is real symplectically similar to

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boxplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boxplus \begin{bmatrix} x & -1 \\ 1 + x^2 & -x \end{bmatrix} \right).$$

By Lemma 2.11, $\begin{bmatrix} x & -1 \\ 1 + x^2 & -x \end{bmatrix}$ is real symplectically similar to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Thus $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boxplus \begin{bmatrix} x & -1 \\ 1 + x^2 & -x \end{bmatrix}$ is real symplectically similar to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boxplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which is a commutator of real symplectic involutions by Lemma 3.2(3) with $\theta = \pi/2$. Thus $A = (-I_2) \boxplus B$ is a product of two commutators of real symplectic involutions. \square

Proof of Theorem 1.1. By Lemma 3.10, we know that each matrix which is similar to $-I_2 \boxplus B$ for $-I_2 \neq B \in \text{Sp}(2, \mathbb{R})$ and $\text{tr}B \geq -2$, is a product of three commutators of real symplectic involutions. For other 4×4 real symplectic matrices, by Lemmas 2.4, 3.4, 3.5, 3.6, 3.7, 3.3, 3.8, 3.9 and 3.11, we get the conclusion. \square

LEMMA 3.12. *Let $n > 1$ and $v \in \mathbb{R}^{2n} \neq 0$. Every positive transvection $T = I_{2n} + \alpha v v^T J_{2n}$ for some positive real number α is a product of two commutators of real symplectic involutions.*

Proof. By Theorem 9 of [1], there exists $P \in \text{Sp}(2n, \mathbb{R})$ such that $P(\sqrt{\alpha}v) = e_1$, where e_1 is the first column of I_{2n} . Since $PTP^{-1} = J_2 \boxplus I_{2n-2}$ and $J_2 \boxplus I_2$ is a product of two commutators of real symplectic involutions by Lemma 3.9, we get the conclusion. \square

Proof of Theorem 1.2. By Theorem 20 of [3], every $2n \times 2n$ real symplectic matrix can be expressed as a product of at most $2n + 3$ positive transvections. Then we get the conclusion since each positive transvection is a product of two commutators of real symplectic involutions by Lemma 3.12. \square

In [8], it is proved that every complex symplectic matrix of size greater than 2 can be decomposed into a product of at most three commutators of symplectic involutions. In this paper, we get the conclusion that every 4×4 real symplectic matrix is a product of at most three commutators of real symplectic involutions. It is an open problem for the real case and $n > 2$ to determine if the number of factors is also three. It is also an open problem to give a necessary and sufficient condition in terms of the Jordan Form like Lemma 2.4 and 2.5 in [8], for a real symplectic matrix to be a commutator of real symplectic involutions.

Acknowledgement. This work was funded by Scientific Research Project of Beijing Educational Committee (No. KM202110028004). The author is grateful to the referee for the helpful comments which improved the paper.

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(Received November 25, 2020)

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