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STRONG L–LIMITED SETS OF ORDER p IN THE DUAL OF BANACH LATTICES

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Abstract. The class of strong L-limited sets of order p in the dual of a Banach lattice is introduced and applied to provide some operator characterizations of Banach lattices with the strong limited p-Schur property and the weak DP* property of order p. Close connections with the strong L-limited sets of order p, with the (V)-sets and with the relatively (weakly) compact sets are established. The disjoint weak DP* property of order p and the positive strong limited p-Schur property are thoroughly considered which can be characterized by disjoint almost limited p-convergent operators.

1. Notation and preliminaries

Limited sets and Dunford-Pettis (DP) sets in a Banach space have interesting applications on the geometry of a Banach space, and in particular on operator spaces [15]. A norm bounded subset C of a Banach space X is called *limited* (resp. DP), if every weak*-null (resp. weakly null) sequence $(x_n^*) \subset X^*$ converges uniformly to zero on C; that is, $\sup_{x\in C} |x_n^*(x)| \to 0$. Based on the relationship between them with relatively (weakly) compact sets, some properties are obtained for a Banach space [17, 19].

We know that every relatively compact set in a Banach space is limited and so DP. If each limited (resp. DP) set in X is relatively compact, then we say that X has the *Gelfand-Phillips (GP)* property (resp. *relatively compact DP property* ($DP_{rc}P$)). Also, if each relatively weakly compact set in X is limited (resp. DP), then we say that X has the *DP** (resp. *DP*) property [17, 19, 15].

There is a close relationship between the weak topology, norm topology and weak* topology in terms of sequences in a Banach space. The space X has the *Schur* (resp. *Grothendieck*) property if each weakly null sequence in X is norm null (resp. each weak*-null sequence in X^* is weakly null). In a Grothendieck space, limited sets and DP sets will be the same.

Like limited sets and DP sets, using disjoint sequences in the dual of a Banach lattice, the class of almost limited (resp. almost DP) sets have been introduced and studied. A norm bounded subset C of a Banach lattice E is *almost limited* (resp.

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almost DP), if every disjoint weak*-null (resp. disjoint weakly null) sequence $(x_n^*) \subset E^*$ converges uniformly to zero on C. A Banach lattice E has the *strong GP* property (resp. *strong* $DP_{rc}P$) if each almost limited (resp. almost DP) set in E is relatively compact. The weak DP* property and weak DP property can be considered on Banach lattices. E has the *weak DP** (resp. *weak DP*) property if each relatively weakly compact set in E is almost limited (resp. almost DP) [2, 7, 8, 13, 35]. A norm bounded subset A of a Banach lattice E is an L*-weakly compact* set, if every disjoint sequence in $sol(A)$ is norm null, where $sol(A) = \{y \in E : |y| \leqslant |x| \text{ for some } x \in A\}$ denotes the solid hull of A. Every L-weakly compact set is almost limited and the converse holds in a Banach lattice with order continuous norm. The connection between L-weakly compact sets and almost limited sets is considered in [13].

Later, by positive weak*-null sequences in the dual of a Banach lattice, positively limited sets have been introduced. A norm bounded subset $C \subset E$ is *positively limited*, if every positive weak*-null sequence $(x_n^*) \subset E^*$ converges uniformly to zero on C. It was proved that a norm bounded subset $C \subset E$ is almost DP if and only if every positive weakly null sequence $(x_n^*) \subset E^*$ converges uniformly to zero on C. According to the relationship between positively limited, almost limited, limited and relatively (weakly) compact sets, some properties on a Banach lattice such as positive DP* property and positive GP property are obtained in [5].

From [5, 13] we see that the following implications are always true in a Banach lattice and that their converses do not hold in general:

$$\begin{array}{ccc} L\text{-weakly compact} & \Longrightarrow & \text{almost Dunford–Pettis} \\ \Downarrow & & \Uparrow \\ \text{almost limited} & \Longrightarrow & \text{positively limited} \end{array}$$

Like the class of sets we have in a Banach space, classes of sets in the dual of that space have also been introduced and studied. For the first time, L-sets have been introduced by Leavelle in his paper [25]. The class of L-sets in X^* is defined in this way: A norm bounded set $C \subseteq X^*$ is called an L*-set*, if every weakly null sequence $(x_n) \subset X$ converges uniformly to zero on C. It can be easily shown that, a Banach space X has the Schur property, if and only if B_{X^*} is an L-set. Also, X does not contain a copy of ℓ_1 (resp. X has the RDP* property) if and only if each L-set in X^* is relatively compact (resp. relatively weakly compact). Several consequences concerning limited and DP sets were obtained in [18, 25].

Due to the interesting applications of L-sets on the geometry of Banach spaces and operator spaces, in the last few years, using limited and DP sets two classes of L-limited, and L-DP sets have been introduced in the dual of a Banach space. If $C \subseteq X^*$ is a norm bounded set, and every limited weakly null (resp. DP weakly null) sequence $(x_n) \subset X$ converges uniformly to zero on C, then we say that C is an L*-limited* (resp. L*-DP*) set. Each relatively weakly compact set in the dual of a Banach space is L-DP and so L-limited. If the converse is also true, then we say that X has the L-limited (resp. L-DP) property. A Banach space X has the GP property (resp. $DP_{rc}P$), if and only if B_{X^*} is an L-limited (resp. L-DP) set. Each L-set is an L-DP set and so an

L-limited set. Based on the properties of these sets, some operator characterizations of the GP property, DP* property, $DP_{rc}P$ and DP property have been obtained in [29, 36].

Two classes of limited completely continuous operators and DP completely continuous operators are also introduced in [30, 36]. An operator $T : X \to Y$ between two Banach spaces is called *limited completely continuous (abb. lcc)* (resp. *DP completely continuous (abb. DPcc)*) if for every limited (resp. DP) weakly null sequence $(x_n) \in X$, $\|Tx_n\| \to 0$. It is proved that X has the L-limited property (resp. L-DP property) if and only if each lcc (resp. DPcc) operator from X into ℓ_∞ is weakly compact. Each weakly compact operator is DPcc and so lcc.

Recently the class of strong L-limited sets in the dual of a Banach lattice was introduced by [24] and some results were obtained. A norm bounded subset $A \subset E^*$ is a *strong L-limited set* if for every weakly null and almost limited sequence (x_n) of E, $\sup_{f\in A} |f(x_n)| \to 0$. These sets are closely related to the strong GP property and *almost limited completely continuous (abb. alcc)* operators which were studied in the paper [7]. Recall that an operator $T : E \to Y$ is called alcc if for every almost limited weakly null sequence $(x_n) \in E$, $\|Tx_n\| \to 0$. It is clear that an operator $T : E \to X$ is alcc if and only if $T^*(B_{X^*})$ is a strong L-limited set in E^*. Each weakly compact operator is lcc [30], but the identity operator on each reflexive non-discrete Banach lattice such as $Id_{L^2(-\pi,\pi)}$ is a weakly compact operator, which is not alcc. The identity operator on each non-reflexive Banach lattice with the strong GP property such as Id_{ℓ_1} is alcc, but it is not weakly compact.

Unlike L-limited sets, and L-DP sets, there is no connection between strong L-limited sets and relatively weakly compact sets, in general. In [24], it has been shown that a strong L-limited set in the dual of a Banach lattice E is relatively weakly compact if and only if each alcc operator from E into ℓ_∞ is weakly compact. Also, a relatively weakly compact set in the dual of a Banach lattice E is a strong L-limited set if and only if each weakly compact operator from E into c_0 is alcc.

The interest in studying p-versions of geometric properties of Banach spaces and Banach lattices has increased in the last years. The p-GP property in spaces of operators and DP type like sets were studied in Banach spaces in [22]. A Banach space X has the p-GP property if every limited weakly p-summable sequence in X is norm null. A norm bounded subset A of X is a weakly p-DP set if each weakly p-summable sequence in X^* converges uniformly to zero on A. Also, a norm bounded subset B of X^* is a weakly p-L-set (resp. an L_p-limited set) if each weakly p-summable (resp. limited weakly p-summable) sequence in X converges uniformly to zero on A. Sufficient conditions for the p-GP property of some spaces of operators in terms of the limited p-convergence of the evaluation operators were obtained in [22]. Because strong L-limited sets of order p are closely related to the strong limited p-Schur property and *almost limited p-convergent (abb. alpc)* operators which were studied in the paper [9, 10], we decided to study them to derive more applications of alpc operators on a Banach lattice. To introduce strong L-limited sets of order p, instead of almost limited weakly null sequences, almost limited weakly p-summable sequences are used. Recall that for each $1 \leqslant p < \infty$, a sequence (x_n) of a Banach space X is called *weakly p-summable* if for each $x^* \in X^*$, $(x^*(x_n)) \in \ell_p$ and (x_n) is said to be *weakly p-convergent* to an $x \in X$ if the sequence $(x_n - x) \in \ell_p^w(X)$, where $\ell_p^w(X)$ is the space of all weakly p-summable sequences in X. The weakly ∞-convergent sequences are simply the weakly convergent sequences. Recent developments can be found in [9, 10, 12, 38]. Some useful and inspiring concepts of the properties which will be considered in this article are collected as follows:

A Banach space X has the:

- DP* property of order p (*abbr.* p-DP* property) if each weakly p-summable sequence in X is limited [38].
- Schur property of order p (*abbr.* p-Schur property) if each weakly p-summable sequence in X is norm null [38].
- limited p-Schur property (or p-GP property) if each limited weakly p-summable sequence in X is norm null [22, 38].

A Banach lattice E has the:

- weak DP* property of order p (*abbr.* p-weak DP* property) if each weakly p-summable sequence in E is almost limited [9].
- positive Schur property of order p (*abbr.* p-positive Schur property) if each weakly p-summable sequence in E^+ is norm null [38].
- strong limited p-Schur property if each almost limited weakly p-summable sequence in E is norm null [9].

Throughout this paper E,F are Banach lattices, X,Y are Banach spaces. B_E is the closed unit ball of E. The lattice operations are weakly sequentially continuous in E, if for every weakly null sequence $(x_n) \subset E$, $|x_n| \xrightarrow{w} 0$. Also, the lattice operations are weak* sequentially continuous in E^*, if for every weak*-null sequence $(x_n^*) \subset E^*$, $|x_n^*| \xrightarrow{w^*} 0$ [1, 27].

An operator $T: E \to X$ is called

- *disjoint p-convergent*, if for every disjoint weakly p-summable sequence $(x_n) \in E$, $\|Tx_n\| \to 0$ [38].
- *limited p-convergent*, if for every limited weakly p-summable sequence $(x_n) \in E$, $\|Tx_n\| \to 0$ [22, 38].
- *almost DP*, if for every disjoint weakly null sequence $(x_n) \in E$, $\|Tx_n\| \to 0$ [2].
- *almost limited p-convergent (alpc)*, if for every weakly p-summable almost limited sequence $(x_n) \in E$, $\|Tx_n\| \to 0$ [10].
- *M-weakly compact*, if for every disjoint sequence $(x_n) \in B_E$, $\|Tx_n\| \to 0$ [1, 27].
- *order weakly compact*, if for every order bounded disjoint sequence $(x_n) \in E$, $\|Tx_n\| \to 0$ [1, 27].

In [23], using limited weakly p-summable sequences in Banach spaces, the class of L-limited sets of order p (L_p-limited sets) in the dual of Banach spaces have been introduced and studied. In this paper "strong L-limited sets of order p in the dual of Banach lattices" are introduced to develop the concept of L_p-limited sets in the dual of Banach lattices. Several operator characterizations, relating these sets to the strong limited p-Schur property, the p-weak DP* property, and almost limited p-convergent operators are provided. Connections with (V)-sets, L_p-limited sets, and relatively weakly compact sets are also explored. The paper further introduces the positive strong limited p-Schur property and studies disjoint almost limited p-convergent operators. Motivated by the works mentioned above, the present paper is organized as follows: In section 2, the p-version of strong L-limited sets (or strong L-limited sets of order p) are introduced. One of our aims in this section is to consider the behavior of the solid hull of strong L-limited sets of order p in the dual of Banach lattices. In section 3, we continue the study of the strong L-limited sets of order p to obtain the relationships between them with the (V)-sets, with the L_p-limited sets, with the strong L-limited set and with the relatively (weakly) compact sets. Moreover, with respect to almost limited p-convergent operators some operator characterizations of Banach lattices with the strong limited p-Schur property and the p-weak DP* property are obtained that shall be useful later. In particular, it is proved that each limited p-convergent operator on a Banach lattice is unconditionally convergent. In the last section of the present paper, the concept of positive strong limited p-Schur property and the class of disjoint almost limited p-convergent operators are studied. In addition, where limited p-convergent operators, disjoint almost limited p-convergent operators and almost limited p-convergent operators are the same, will be considered. Also, the disjoint p-weak DP* property is defined by replacing weakly p-summable sequences in the definition of the p-weak DP* property with disjoint weakly p-summable sequences. We show that under some conditions in a Banach lattice with the disjoint p-weak DP* property, the class of disjoint almost limited p-convergent operators and disjoint p-convergent operators will be the same.

2. Strong L-limited sets of order p

By replacing weakly null sequences in the definition of strong L-limited sets with the weakly p-summable sequences, strong L-limited sets of order p are defined.

DEFINITION 2.1. A norm bounded subset $A \subset E^*$ is a *strong L-limited set of order p (abbr. a strong L_p-limited set)* if for every weakly p-summable and almost limited sequence (x_n) of E, $\sup_{f\in A}|f(x_n)| \to 0$.

It can be proved that $A \subset E^*$ is a strong L_p-limited set if and only if for each sequence (f_n) in A and each weakly p-summable and almost limited sequence (x_n) of E, $f_n(x_n) \to 0$. The sequence (x_n) is almost limited, if $\{x_n : n \in \mathbb{N}\}$ is an almost limited set. The following theorem provides an operator characterization of a strong L_p-limited set. Recall that an operator $T : X \to E$ is almost limited if for every disjoint weak*-null sequence $(f_n) \subset E^*$, $\|T^* f_n\| \to 0$; that is $T(B_X)$ is an almost limited subset of E.

THEOREM 2.2. *For a bounded subset A of E^* and $1 < p < \infty$, the following assertions are equivalent.*

(a) A is a strong L_p-limited set,

(b) $T^*(A)$ is relatively compact for every almost limited operator $T : \ell_{p^*} \to E$.

Proof. $(a) \Rightarrow (b)$ Let A be a strong L_p-limited set of E^* and (f_n) be a sequence in A. Let $T : \ell_{p^*} \to E$ be an almost limited (weakly p-compact, since $B_{\ell_{p^*}}$ is weakly p-compact [12]) operator. Define $S : E \to \ell_\infty$ by $Sx = (f_n(x))$, $x \in E$. Let (x_n) be an almost limited weakly p-summable sequence in E. Since A is a strong L_p-limited set of E^*, $\|Sx_n\| = \sup_i |f_i(x_n)| \to 0$, as $n \to \infty$ and thus S is alpc. Then by [10, Theorem 2.4], $ST : \ell_{p^*} \to \ell_\infty$ is compact. Since T^*S^* is compact, $(T^*(f_n)) = (T^*S^*(e_n^*))$ is relatively compact. Hence $T^*(A)$ is relatively compact.

$(b) \Rightarrow (a)$ Let (x_n) be an almost limited weakly p-summable sequence in E. Let $T : \ell_{p^*} \to E$ be an operator such that $T(e_n) = x_n$. Since (x_n) is almost limited, T is an almost limited operator. By hypothesis, $T^*(A)$ is relatively compact and so $\sup_{x^* \in A} |x^*(x_n)| = \sup_{x^* \in A} |T^*x^*(e_n)| \to 0$ which implies that A is a strong L_p-limited set. □

We have the following characterization of the strong limited p-Schur property as follows:

COROLLARY 2.3. *For a Banach lattice E and $1 < p < \infty$, the following assertions are equivalent.*

(a) E has the strong limited p-Schur property,

(b) B_{E^*} is a strong L_p-limited set,

(c) every almost limited operator $T : \ell_{p^*} \to E$ is compact.

Note that for a disjoint weak*-null sequence (f_n) in E^*, the sequence $(|f_n|)$ is not necessarily weak*-null in E^*, see [13, Example 2.1]. In order to establish our next result, recall that a Banach lattice E has the *property (d)*, if for each disjoint weak*-null sequence (f_n) in E^*, $|f_n| \xrightarrow{w^*} 0$. Each σ-Dedekind complete Banach lattice has the property (d), see [24].

THEOREM 2.4. *Let $T : E \to F$ be an order bounded operator and F has the property (d). If A is a strong L_p-limited set of F^*, then $T^*(A)$ is a strong L_p-limited set in E^*.*

Proof. Let A be a strong L_p-limited subset of F^* and $T : E \to F$ be an order bounded operator. Then for each weakly p-summable and almost limited sequence (x_n) of E, (Tx_n) is a weakly p-summable positively limited sequence in F, see [5, Theorems 2.6 & 2.11]. By the property (d) of F and [5, Theorems 2.7], (Tx_n) is an almost limited sequence in F. Hence $\sup_{f\in A}|(T^*f)(x_n)| = \sup_{f\in A}|f(Tx_n)| \to 0$. This implies that $T^*(A)$ is a strong L_p-limited set in E^*. $\square$

Note that order boundedness of the operator T cannot be removed in Theorem 2.4.

EXAMPLE 2.5. Let $(r_n(t))_{n=1}^{\infty}$ denote the sequence of Rademacher functions on $[0,1]$. Consider $S: L_2[0,1] \to c_0$ defined by

$$S(f) = \left(\int_0^1 f(t)r_n(t)\right) \qquad \text{for all } f \in L^2[0,1].$$

Since c_0 has the strong limited p-Schur property, $B_{c_0^*}$ is a strong L_p-limited subset of ℓ_1, but $S^*B_{c_0^*}$ is not a strong L_p-limited set in $L_2[0,1]$. Indeed, $S^*(e_n^*) = r_n$, for all $n \in \mathbb{N}$ where (e_n^*) is the canonical basis of ℓ_1 and (r_n) is not a strong L_p-limited set in $L_2[0,1]$, for all $p \geqslant 2$.

EXAMPLE 2.6. Note that an order interval in the dual of a Banach lattice is not necessarily a strong L_p-limited set. It follows immediately from the equality $\sup_{g\in[-f,f]}|g(x_n)| = f(|x_n|)$, that for each $f \in (E^*)^+, [-f,f]$ is a strong L_p-limited set if and only if for every weakly p-summable and almost limited sequence (x_n) of E, $|x_n| \xrightarrow{w} 0$. The Rademacher sequences (r_n) in $L^1[0,1]$ are weakly 2-summable and so weakly p-summable for each $p \geqslant 2$ (cf. [28, Proposition 3.6]), and almost limited (since $L^1[0,1]$ has the p-weak DP* property), but $|r_n| = 1$ for all n. This implies that an order interval in $(L^1[0,1])^*$ is not necessarily a strong L_p-limited set.

Recall that a Banach lattice E is weak p-consistent if for every weakly p-summable sequence (x_n) in E, $(|x_n|)$ is weakly p-summable [38, Definition 3.3.5].

DEFINITION 2.7. A Banach lattice E is *almost limited weak p-consistent* if for every weakly p-summable and almost limited sequence (x_n) in E, $(|x_n|)$ is weakly p-summable almost limited in E.

Each Banach lattice with the strong limited p-Schur property is almost limited weak p-consistent. The converse is false. For instance, ℓ_∞ is weak p-consistent and so almost limited weak p-consistent (since, B_{ℓ_∞} is almost limited), but ℓ_∞ does not have the strong limited p-Schur property. $L^2[0,1]$ is not almost limited weak p-consistent. In fact, the Rademacher sequence $(r_n(t))_{n=1}^{\infty}$ is weakly p-summable (cf. [28, Proposition 3.6]) and almost limited (cf. [26, Proposition 2.3]) in $L^2[0,1]$ for each $p \geqslant 2$, but $|r_n| = 1$ for all n. Note that since $L^2[0,1]$ has order continuous norm, its order intervals are almost limited and $-\mathbf{1} \leqslant r_n \leqslant \mathbf{1}$, for all n.

THEOREM 2.8. *Let A be a norm bounded subset of E^*. Suppose that E is almost limited weak p-consistent. Then the following are equivalent:*

(a) A is a strong L_p-limited set,

(b) $sol(A)$ *is a strong* L_p*-limited set,*

(c) $|A|$ *is a strong* L_p*-limited set.*

Proof. $(a) \Rightarrow (b)$. Assume by way of contradiction that $A \subset E^*$ is a strong L_p-limited set, and $sol(A)$ is not a strong L_p-limited set. Then there exist a sequence (f_n) in $sol(A)$, a weakly p-summable and almost limited sequence (x_n) of E and an $\varepsilon > 0$ such that $|f_n(x_n)| \geqslant \varepsilon$ for all n. For each n there exists $g_n \in A$, such that $|f_n| \leqslant |g_n|$. Since E is almost limited weak p-consistent, the sequences $(|x_n|)$ is weakly p-summable and almost limited. However, $\varepsilon \leqslant |f_n(x_n)| \leqslant |f_n|(|x_n|) \leqslant |g_n|(|x_n|)$. Since $|g_n|(|x_n|) = \sup\{|g_n(y)| : |y| \leqslant |x_n|\}$, for every n there is a sequence (y_n) in E with $|y_n| \leqslant |x_n|$ and $|g_n(y_n)| > \varepsilon$. Hence $(|y_n|)$ and so (y_n) itself is weakly p-summable and almost limited sequence. Since A is a strong L_p-limited set, $g_n(y_n) \to 0$ which is impossible. Hence, $sol(A)$ is a strong L_p-limited set.

$(b) \Rightarrow (c)$. It is clear, since $|A| \subset sol(|A|) = sol(A)$.

$(c) \Rightarrow (a)$. Let $|A|$ be a strong L_p-limited set in E, then $sol(|A|) = sol(A)$ is a strong L_p-limited set in E. Since $A \subset sol(A)$, A is a strong L_p-limited set in E. □

To conclude this section, note that each disjoint sequence in the solid hull of a weakly null sequence in a Banach lattice is weakly null, [1, Theorem 13.3]. However, in [20, Theorem 3.6], it was proved that every disjoint sequence in the solid hull of a weakly p-summable sequence is weakly p-summable, for all $p \geqslant 2$. So we can prove the following theorem.

THEOREM 2.9. *Suppose that E has the property (d) and A is a strong L_p-limited set in E^*. Then for each $p \geqslant 2$, the following assertions hold:*

(a) *If (f_n) is a sequence in $sol(A)$ satisfying $|f_n| \xrightarrow{w^*} 0$, then (f_n) is a strong L_p-limited set.*

(b) *If (f_n) is a disjoint sequence in $sol(A)$, then $|f_n| \xrightarrow{w^*} 0$ and (f_n) is a strong L_p-limited set.*

Proof. (a) To prove that (f_n) is a strong L_p-limited set in E^*, it suffices to show that $f_n(x_n) \to 0$ for each weakly p-summable almost limited sequence (x_n) in E. Otherwise, passing to a subsequence if necessary, there would exist a weakly p-summable almost limited sequence (x_n) in E with $|f_n|(|x_n|) > |f_n(x_n)|$ for some $\varepsilon > 0$ and for all n. Since $|f_n| \xrightarrow{w^*} 0$, similar to [37, Lemma 2.2] we can find a strictly increasing subsequence (n_k) and a disjoint sequence (y_m) in E such that $|f_{n_{m+1}}|(y_m) > \frac{\varepsilon}{2}$ for all sufficiently large m. There is a sequence (g_m) in A and a disjoint sequence (u_m) in E with $|u_m| \leqslant y_m \leqslant |x_{n_{m+1}}|$ and $|g_{m+1}(u_m)| > \frac{\varepsilon}{2}$ for all sufficiently large m. Since (u_m) is a disjoint sequence in the solid hull of a weakly p-summable sequence $(x_{n_{m+1}})$, by [20, Theorem 3.6], (u_m) is weakly p-summable and also it is almost limited by the property (d). Since A is a strong L_p-limited set in E^*, $|g_{m+1}(u_m)| \leqslant \sup_{f \in A} |f(u_m)| \to 0$, which is impossible.

(b) We know that each order bounded disjoint sequence in a Banach lattice with the property (d) is weakly p-summable and almost limited [9, 13]. Also, the solid hull of an almost limited set in a Banach lattice with the property (d) is almost limited, see [24, Lemma 2.1]. Hence for each $x \in E^+$, each disjoint sequence of $[0,x]$ is weakly p-summable and almost limited and so converges uniformly to zero on A. Hence by [1, Theorem 4.40], for each $\varepsilon > 0$ there exist some $0 \leqslant g \in E^*$ such that $(|f|-g)^+(x) < \frac{\varepsilon}{2}$, for all $f \in A$. Now, let (f_n) be disjoint sequence in $sol(A)$ and pick a sequence (g_n) in A with $|f_n| \leqslant |g_n|$ for all n. Also note that since $(|f_n| \wedge g)$ is an order bounded disjoint sequence of E^*, $(|f_n| \wedge g)(x) \to 0$. Then there is $N \in \mathbb{N}$ such that for all $n \geqslant N$, $(|f_n| \wedge g)(x) \leqslant \frac{\varepsilon}{2}$ Therefore

$$|f_n|(x) = (|f_n|-g)^+(x) + (|f_n| \wedge g)(x) \leqslant (|g_n|-g)^+(x) + (|f_n| \wedge g)(x) \leqslant \varepsilon.$$

Since $\varepsilon > 0$ be arbitrary, $|f_n|(x) \to 0$ and so $|f_n| \xrightarrow{w^*} 0$. Now, it follows from (a) that (f_n) is a strong L_p-limited set. $\square$

Note that the property (d) in Theorem 2.9 cannot be removed. In fact, Banach sequence space c has the strong limited p-Schur property and so the closed unit ball B_{c^*} is a strong L_p-limited set, while for a disjoint sequence (f_n) in c^* defined by $f_n = (0,0,\ldots,1_{2n},-1_{2n+1},0,0,\ldots)$ for all n, we have $|f_n|(\mathbf{1}) = \sup_{x\in[-\mathbf{1},\mathbf{1}]} |f_n(x)| = \sup_{x\in B_c} |f_n(x)| = \|f_n\| = 2$, where $\mathbf{1} = (1,1,1,\ldots) \in c$.

3. Strong L_p-limited sets and spaces of operators

In this section by the relationship between strong L_p-limited sets, (V)-sets, L_p-limited sets and relatively (weakly) compact sets, some operator characterizations of Banach lattices with the strong limited p-Schur property and the weak DP* property of order p are obtained with respect to almost limited p-convergent operators. The following assertions can be easily proved:

- E has the strong limited p-Schur property if and only if B_{E^*} is a strong L_p-limited set.
- E has the limited p-Schur property if and only if B_{E^*} is an L_p-limited set.
- E has the strong GP property if and only if B_{E^*} is a strong L-limited set.
- E has the GP property if and only if B_{E^*} is an L-limited set.

First, we see that the following implications are always true in a Banach lattice and that their converses do not hold in general:

$$\begin{array}{ccc} \text{strong } L\text{-limited set} & \Longrightarrow & \text{strong } L_p\text{-limited set} \\ \Big\Downarrow & & \Big\Uparrow \\ L\text{-limited set} & \Longrightarrow & L_p\text{-limited set} \end{array}$$

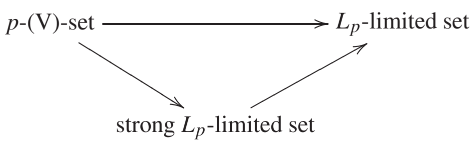
EXAMPLE 3.1. The converse of none of the implications in the diagram above is true, in general.

- $L^1[0,1]$ has the GP property, but it does not have the strong GP property. Hence, $B_{L^\infty[0,1]}$ is an L-limited set while it is not a strong L-limited set.

- $L^2[0,1]$ has the limited p-Schur property, but it does not have the strong limited p-Schur property. Hence, $B_{L^2[0,1]}$ is an L_p-limited set while it is not a strong L_p-limited set.

- $C[0,1]$ has the strong limited p-Schur property, but it does not have the strong GP property. Hence $B_{(C[0,1])^*}$ is a strong L_p-limited set while it is not a strong L-limited set.

A bounded subset A of X^* is called a (V)*-set* if $\sup_{x^*\in A}|\langle x_n, x^*\rangle| \to 0$, for every w.u.c. series $\sum x_n$ in X. From [22], a subset A of a Banach space X^* is called a *V-set of order p (or a weakly p- L-set)* if every weakly p-summable sequence (x_n) in X converges uniformly to zero on A. A Banach space X has the p-Schur property if and only if B_{X^*} is a p-(V)-set. The relationship between strong L_p-limited sets, L_p-limited sets and p-(V)-sets are considered as follows.



EXAMPLE 3.2. For the converse of implications, see the following examples.

- Each non-KB-space with order continuous norm such as c_0 has the limited p-Schur property, but it does not have the p-Schur property. Hence, B_{ℓ_1} is an L_p-limited set while it is not a p-(V)-set.

- Each discrete non-KB-space with order continuous norm such as c_0 has the strong limited p-Schur property, but it does not have the p-Schur property. Hence, B_{ℓ_1} is a strong L_p-limited set while it is not a p-(V)-set.

We know that for each Banach lattice with the limited p-Schur property and without the 1-Schur property, such as $c, c_0, C[0,1]$, the closed unit ball of the dual is an L_p-limited set which is not a (V)-set. The following theorem shows that the converse is true in general that shall be useful later a couple of times.

THEOREM 3.3. *Every (V)-set in the dual of a Banach lattice is an L_p-limited set.*

Proof. Step 1. First note that for each limited weakly p-summable sequence (x_n) in a Banach lattice E, $|x_n| \xrightarrow{w} 0$. Indeed, assume by way of contradiction that $(x_n)_n$ is a limited weakly p-summable sequence $(x_n)_n$ in E, but $|x_n| \overset{w}{\nrightarrow} 0$. Hence there exists $0 \leqslant f \in E^*$ satisfying $f(|x_n|) > \varepsilon$ for some $\varepsilon > 0$ and for all n. As in the proof of Theorem 2 of [32], we can find a sequence $(f_n) \subset [-f, f]$ such that $f_n \xrightarrow{w^*} 0$ and $f_n(x_n) \geqslant \varepsilon$ for all n. But (x_n) is a limited sequence in E and so this is impossible.

Step 2. Now, we show that every (V)-set in the dual of a Banach lattice E is an L_p-limited set. Let A be a (V)-set in E^* and $(x_n) \subset E$ be a limited weakly p-summable sequence $(x_n)_n \subset E$. It is sufficient to show that $\sup_{f\in A} |f(x_n)| \to 0$. For this, note that from [9], for each $x \in E^+$, each disjoint sequence of $[0, x]$ is weakly p-summable and so converges uniformly to zero on A. Hence by [1, Theorem 4.40], for each $\varepsilon > 0$ there exist some $0 \leqslant g \in E^*$ such that $(|f| - g)^+(x) < \varepsilon$, for all $f \in A$. Also, $|x_n| \xrightarrow{w} 0$ and so $g(|x_n|) \to 0$. Therefore

$$|f(x_n)| \leqslant |f|(|x_n|) = (|f| - g)^+(|x_n|) + (|f| \wedge g)(|x_n|) \leqslant \varepsilon + g(|x_n|) \leqslant 2\varepsilon.$$

Since $\varepsilon > 0$ and $f \in A$ are arbitrary, $\sup_{f\in A} |f(x_n)| \to 0$ which proves that A is an L_p-limited set. $\square$

Therefore, we can draw the following diagram:

$$\begin{array}{ccc} p\text{-(V)-set} & \Longrightarrow & \text{strong } L_p\text{-limited set} \\ \Downarrow & & \Uparrow \\ (V)\text{-set} & \Longrightarrow & L_p\text{-limited set} \end{array}$$

THEOREM 3.4. *If E^* has the weak* sequentially continuous lattice operations, then each L_p-limited set in E^* is a strong L_p-limited set. The converse is also true, if each order interval in E is weakly p-compact.*

Proof. Just notice that if E^* has the weak* sequentially continuous lattice operations, then each almost limited set in E is a limited set [5, Theorem 2.5]. This implies that each L_p-limited set in E^* is a strong L_p-limited set.

For the converse, if each L_p-limited set in E^* is a strong L_p-limited set, we show that E^* has the weak* sequentially continuous lattice operations. From [5, Theorem 2.5] it is enough to show that each operator $T : E \to c_0$ is AM-compact. If $T : E \to c_0$ is an operator, then T is limited p-convergent and so $T^*(B_{\ell_1})$ is an L_p-limited set. By hypothesis, $T^*(B_{\ell_1})$ is a strong L_p-limited set and then T is alpc. For each $x \in E^+$, an order interval $[-x, x]$ is positively limited and by hypothesis, weakly p-compact in E. By [10], $T[-x, x]$ is relatively compact and so $T : E \to c_0$ is AM-compact. This implies that E^* has weak*-sequentially continuous lattice operations. $\square$

The converse of Theorem 3.4 is false in general. Consider ℓ_∞. By the p-weak DP* property, each strong L_p-limited set in ℓ_∞^* is a p-(V)-set, and so it is a strong L_p-limited set. But ℓ_∞^* does not have the weak* sequentially continuous lattice operations. We continue this section stating some useful operator characterizations of the properties considered in this article. The identity operator on a Banach lattice E is:

- p-convergent (*abbr.* pc) if and only if E has the p-Schur property.
- unconditionally convergent (*abbr.* uc) if and only if E has the 1-Schur property.
- limited p-convergent (*abbr.* lpc) if and only if E has the limited p-Schur property.
- almost limited p-convergent (*abbr.* alpc) if and only if E has the strong limited p-Schur property.

From Theorem 3.3, relations between alpc operators, lpc operators, uc operators and pc operators are summarized as follows:

$$\begin{array}{ccc} \text{pc} & \Longrightarrow & \text{uc} \\ \Downarrow & & \Uparrow \\ \text{alpc} & \Longrightarrow & \text{lpc} \end{array}$$

The set of all p-convergent (resp. alpc) operators from E into X is denoted by $\mathscr{L}_{pc}(E,X)$ (resp. $\mathscr{L}_{alpc}(E,X)$). The p-weak DP* property can be characterized by means of alpc operators as follows:

THEOREM 3.5. *For a Banach lattice E, the following are equivalent:*
(a) *E has the p-weak DP* property.*
(b) *Every strong L_p-limited subset of E^* is a p-(V)-set.*
(c) *For every Banach space X, $\mathscr{L}_{pc}(E,X) = \mathscr{L}_{alpc}(E,X)$.*
(d) $\mathscr{L}_{pc}(E,c_0) = \mathscr{L}_{alpc}(E,c_0)$.

Proof. (a)$\Rightarrow$(b) Let $B \subset E^*$ be a strong L_p-limited set. If (x_n) is a weakly p-summable sequence in E, the assumption yields that (x_n) is an almost limited sequence in E, and hence $\sup_{x^* \in B} |x^*(x_n)| \to 0$. Thus B is a p-(V)-set.

(b) $\Rightarrow$ (c) Let X be an arbitrary Banach space. If $T : E \to X$ is an alpc operator, then $T^*(B_{X^*})$ is a strong L_p-limited set in E^*. By the assumption, $T^*(B_{X^*})$ is a p-(V)-set, and hence T is a p-convergent operator.

(c)$\Rightarrow$(d) Obvious.

(d) $\Rightarrow$ (a) We show that for each weakly p-summable sequence $(x_n)_{n=1}^{\infty}$ in E, and disjoint weak*-null sequence (x_n^*) in E^*, $x_n^*(x_n) \to 0$. From [21, Proposition 2.3], the operator $T : E \to c_0$ defined by $Tx = (f_n(x))$, $x \in E$ is alpc and by hypothesis, it is p-convergent. Hence $|f_n(x_n)| \leqslant \|Tx_n\| \to 0$ which implies that E has the p-weak DP* property. $\square$

COROLLARY 3.6. *For a Banach lattice E with the property (d), the following assertions are equivalent:*

(a) E has the p-weak DP* property,

(b) each positive operator $T : E \to F$ is p-convergent, for each Banach lattice F with the strong limited p-Schur property and property (d).

Proof. $(a) \Rightarrow (b)$ Let $T : E \to F$ be a positive operator between two Banach lattices E and F such that F has the strong limited p-Schur property. It follows from [5, Theorems 2.7 & 2.11] that T takes each almost limited set A in E to an almost limited set in F. By [5, Theorem 3.6], each almost limited set in F is relatively compact. Hence for each almost limited set A in E, the set $T(A)$ is a relatively compact set in F and so it is alpc. Then $T^*(B_{F^*})$ is a strong L_p-limited set, and by Theorem 3.5 it is a p-(V)-set, and hence T is a p-convergent operator.

$(b) \Rightarrow (a)$ Assume that E does not have the p-weak DP* property. Then there are a disjoint weak*-null sequence (x_n^*) in E^*, and a weakly p-summable sequence (x_n) in E, and some $\varepsilon > 0$ with $|x_n^*(x_n)| > \varepsilon > 0$, for all n. Since F has the property (d), $(|x_n^*|)$ is weak* null in E^*, and so we may assume that (x_n^*) is positive. Hence the positive operator $T : E \to c_0$ defined by $T(x) = (x_n^*(x))$ for all $x \in E$, is not p-convergent, and this is a contradiction. □

Note that by corollary 3.6, the positivity of the operator T cannot be removed. Although, $L^1[0,1]$ has the p-weak DP* property, and c_0 has the strong limited p-Schur property and property (d), but an operator $T : L^1[0,1] \to c_0$ defined as

$$Tf = \left(\int_0^1 f(t) r_n(t) \mathrm{d}t \right) \qquad \text{for all} \quad f \in L^1[0,1],$$

where $r_n(t)$ is the nth Rademacher function on $[0,1]$, is not p-convergent. Indeed, $(r_n(t))_{n=1}^{\infty}$ is weakly p-summable in $L^1[0,1]$ for all $2 \leqslant p$, but $\|Tr_n\| = 1$, $n \in \mathbb{N}$.

Unlike L-limited sets, which cannot be relatively compact unless the space is finite dimensional, the situation is different for strong L_p-limited sets.

PROPOSITION 3.7. *If B_E is a weakly p-precompact set and E has the p-weak DP* property, then each strong L_p-limited set in E^* is relatively compact.*

Proof. Since the closed unit ball B_E is a weakly p-precompact set, each L_p-set in E^* is relatively compact [22, Corollary 27]. By the p-weak DP* property, each strong L_p-limited set in E^* is an L_p-set and so relatively compact. □

Similar to [29, Theorem 2.2], it can be proved that each disjoint (or positive) weak*-null sequence in the dual of a Banach lattice is a strong L_p-limited set. Just notice that from [21], each disjoint operator $T : E \to c_0$ is alpc.

THEOREM 3.8. *Let E be a Banach lattice. The following are equivalent:*

(a) Each strong L_p-limited set in E^* is relatively compact.

(b) E has the dual Schur property and E does not contain a copy of ℓ_1.

Proof. $(a) \Rightarrow (b)$ To prove that that E has the dual Schur property, it is sufficient to show that B_E is an almost limited set, or equivalently every disjoint operator $T : E \to c_0$ is compact, see [21, Proposition 2.3]. Let $T : E \to c_0$ be a disjoint operator. Then T takes each weakly p-compact almost limited subset of E into a relatively compact set in c_0, and so it is alpc, see [7, 21]. Hence $T^*(B_{\ell_1})$ is a strong L_p-limited set in E^* and by hypothesis (a), it is relatively compact. Hence T^* and so T are compact, which implies that E has the dual Schur property.

For the second part, we know that each L-set in E^* is relatively compact if and only if E does not contain a copy of ℓ_1. Hence if each strong L_p-limited set in E^* is relatively compact, then each L-set in E^* is relatively compact and so E does not contain a copy of ℓ_1.

$(b) \Rightarrow (a)$ Let A be a strong L_p-limited set in E^*. Since E has the dual Schur property, B_E is an almost limited set and so each strong L_p-limited set in E^* is an L-set. On the other hand E does not contain a copy of ℓ_1 and each L-set in E^* is relatively compact. Hence A is relatively compact. $\square$

Note that weakly compact operators and alpc operators are different in general. For instance, $Id_{L^2(-\pi,\pi)}$ is a weakly compact operator which is not alpc. Also, Id_c is an alpc operator which is not weakly compact.

THEOREM 3.9. *For a Banach lattice E, the following are equivalent:*

(a) relatively weakly compact sets and strong L_p-limited sets coincide in E^*,

(b) for each Banach space Y, $\mathscr{L}_{alpc}(E,Y) = \mathscr{L}_w(E,Y)$,

(c) $\mathscr{L}_{alpc}(E,\ell_\infty) = \mathscr{L}_w(E,\ell_\infty)$.

Proof. First we show that each strong L_p-limited set in E^* is relatively weakly compact if and only if each alpc operator from E into each Banach space is weakly compact.

$(a) \Rightarrow (b)$ Suppose that each strong L_p-limited set in E^* is relatively weakly compact and $T : E \to Y$ is alpc. Thus $T^*(B_{Y^*})$ is a strong L_p-limited set. By hypothesis, it is relatively weakly compact and so T is a weakly compact operator.

$(b) \Rightarrow (c)$ It is obvious.

$(c) \Rightarrow (a)$ If there exists a strong L_p-limited set subset A of E^* that is not relatively weakly compact. So there is a sequence $(x_n^*) \subset A$ with no weakly convergent subsequence. An operator $T : E \to \ell_\infty$ defined by $Tx = (\langle x, x_n^* \rangle)$, $x \in E$ is alpc, but it is not weakly compact. Similarly, it is can be proved that each relatively weakly compact set in E^* is a strong L_p-limited set and this finishes the proof. $\square$

As a result of Theorem 3.9, we show that if each strong L_p-limited set in the dual of a Banach lattice is relatively weakly compact, then this property is carried only by its positively complemented sublattices. Consider c_0, c as closed sublattices of ℓ_∞. Since ℓ_∞ is a Grothendieck Banach lattice, each strong L_p-limited set in ℓ_∞^* is relatively weakly compact, but strong L_p-limited sets in $c^* = c_0^* = \ell_1$ cannot be relatively weakly compact. A closed subspace V of E is positively complemented if there is a positive projection P on E whose range is V.

COROLLARY 3.10. *Let F be a positively complemented sublattice of a Banach lattice E and F has the property (d). If each strong L_p-limited set in E^* is relatively* *weakly compact, then every strong L_p-limited set in F^* is relatively weakly compact too.*

Proof. From Theorem 3.9, we must show that each alpc operator $T : F \to \ell_\infty$ is weakly compact. Let $P : E \to E$ be a positive projection whose range is F. Since P is positive and F has the property (d), for each weakly p-summable and almost limited sequence (x_n) in E, (Px_n) is weakly p-summable and almost limited in F. On the other hand, $T : F \to \ell_\infty$ is alpc and so $\|TPx_n\| \to 0$; that is, $TP : E \to \ell_\infty$ is alpc. By hypothesis, each strong L_p-limited set in E^* is relatively weakly compact, and so TP is weakly compact. Hence T is weakly compact. □

A Banach lattice E has the weak Grothendieck property if each disjoint weak*-null sequence in E^* is weakly null. From [34, Proposition 1.3], each Banach lattice with the weak Grothendieck property has the property (d). The Banach lattice c does not have the weak Grothendieck property, but ℓ_1 has the weak Grothendieck property [21]. In general, we can see that in each σ-Dedekind complete Banach lattice E such that E^* has order continuous norm, weak Grothendieck and Grothendieck properties coincide [27, Proposition 1.1.8 & Theorem 5.3.13].

THEOREM 3.11. *Let E be a Banach lattice. The following are equivalent:*

(a) Each strong L_p-limited set in E^* is relatively weakly compact.

(b) E has the weak Grothendieck property and E^* has order continuous norm.

Proof. $(a) \Rightarrow (b)$ If each strong L_p-limited set in E^* is relatively weakly compact, then each L-set in E^* is relatively weakly compact and so by [3, Theorem 3.1], E^* has order continuous norm.

To prove that E has the weak Grothendieck property, it is enough to show that every disjoint operator $T : E \to c_0$ is weakly compact; see [21, Proposition 2.2]. For this, let $T : E \to c_0$ be a disjoint operator. Then by [21, Proposition 2.3], T is alpc and so $T^*(B_{\ell_1})$ is strong L_p-limited set in E^*. By hypothesis (a), $T^*(B_{\ell_1})$ is relatively weakly compact. Hence T^* and so T are weakly compact. This implies that E has the weak Grothendieck property.

$(b) \Rightarrow (a)$ From Theorem 3.9, we must show that each alpc operator on E is weakly compact. From [27, Theorem 5.3.13], we show that alpc operator on E is order weakly compact. Let (x_n) be a disjoint sequence of $[0,x]$, and $x \in E^+$. Then (x_n) is a weakly p-summable, and almost limited sequence in E (by the property (d) of E). Since T is alpc, $\|Tx_n\| \to 0$. This implies that T is order weakly compact and so it is weakly compact. Thus each strong L_p-limited set in E^* is relatively weakly compact. □

It follows from Theorem 3.11 that, each strong L_p-limited set in the dual of a Grothendieck Banach lattice is relatively weakly compact. For the converse see the following proposition. Recall that E has the *Interpolation property (I)*, if for all sequences (x_n) and (y_m) in E such that $x_n \leqslant y_m$, for all m,n, there is an element $u \in E$ satisfying $x_n \leqslant u \leqslant y_m$, for all m,n.

PROPOSITION 3.12. *Suppose that each strong L_p-limited set in E^* is relatively weakly compact. If at least one of the following cases holds, then E is Grothendieck.*

1. *E^* has weak* sequentially continuous lattice operations.*

2. *E has the interpolation property (I).*

Proof. (1) If E^* has weak* sequentially continuous lattice operations, then from [5], each almost limited set in E is limited and so each L_p-limited set in E^* is a strong L_p-limited set. Since by our hypothesis, each strong L_p-limited set in E^* is relatively weakly compact, each L_p-limited set in E^* is relatively weakly compact. To prove that E is Grothendieck, let $T : E \to c_0$ be an operator. Since c_0 has the limited p-Schur property, T is lpc. Then $T^*(B_{\ell_1})$ is an L_p-limited set and so relatively weakly compact in E^*. Hence T^* and so T are weakly compact. Hence E is Grothendieck.

(2) We know that each positive operator $T : E \to c_0$ is alpc and so $T^*(B_{\ell_1})$ is a strong L_p-limited set in E^*. If each strong L_p-limited set in E^* is relatively weakly compact, then T^* and so T are weakly compact. Hence E has the positive Grothendieck property [34]. Hence by the interpolation property (I), and [27, Theorem 5.3.13], E is Grothendieck. □

4. Positive strong limited p-Schur property

In this section the concept of the positive strong limited p-Schur property is considered. Also, with respect to the class of disjoint almost limited p-convergent operators, some operator characterizations are obtained.

DEFINITION 4.1. A Banach lattice E has the *positive strong limited p-Schur property* if each disjoint almost limited sequence $(x_n) \in \ell_p^w(E)$ is norm null.

Each Banach lattice with the strong limited p-Schur property has the positive strong limited p-Schur property, but the converse is false. $L^1[0,1]$ has the positive Schur property and so it has the positive strong limited p-Schur property, while it does not have the strong limited p-Schur property. Now it is a natural question to ask under what conditions these properties will be equivalent?

THEOREM 4.2. *Let E be a Banach lattice such that E^* has the weak* sequentially continuous lattice operations. Then the following assertions are equivalent:*

(a) E has the strong limited p-Schur property,

(b) E has the positive strong limited p-Schur property,

(c) E has the limited p-Schur property.

Proof. $(a) \Rightarrow (b)$ Obvious.

$(b) \Rightarrow (c)$ Since E^* has the weak* sequentially continuous lattice operations, E has the property (d), see [26]. We show that E has order continuous norm: In fact, let

(x_n) be an order bounded disjoint sequence in E. Then (x_n) is a weakly p-summable and almost limited sequence in E, by the property (d). It follows from the positive strong limited p-Schur property that $\|x_n\| \to 0$; that is E has order continuous norm. Hence E has then limited p-Schur property.

$(c) \Rightarrow (a)$ Let A be an almost limited weakly p-compacts subset of E. Then A is positively limited; see [5, Theorem 2.6]. Since E^* has the weak* sequentially continuous lattice operations, by [5, Theorem 2.5], A is limited. By the limited p-Schur property of E, A is relatively compact, and so E has the strong limited p-Schur property. □

We show that it is under the σ-Dedekind completeness that the positive strong limited p-Schur property coincide with the limited p-Schur property:

THEOREM 4.3. *Let E be a σ-Dedekind complete Banach lattice. Then the following assertions are equivalent:*

(a) E has the positive strong limited p-Schur property,

(b) E has the limited p-Schur property.

Proof. $(a) \Rightarrow (b)$ Since ℓ_∞ does not have the positive strong limited p-Schur property, E does not contain a copy of ℓ_∞. On the other hand, E is σ-Dedekind complete and by [27, Corollary 2.4.3], it has order continuous norm. Hence E has the limited p-Schur property.

$(b) \Rightarrow (a)$ Let (x_n) be a disjoint almost limited weakly p-summable sequence in E. Since E is a σ-Dedekind complete Banach lattice, it follows from [5, Theorem 2.8] that (x_n) is limited. Hence by our hypothesis, $\|x_n\| \to 0$ which implies that E has the positive strong limited p-Schur property. □

In order to establish some operator characterizations of the positive strong limited p-Schur property, define the class of disjoint almost limited p-convergent operators as follows:

DEFINITION 4.4. A bounded linear operator $T : E \to X$ from a Banach lattice E into a Banach space X is disjoint almost limited p-convergent (*abbr.* dalpc) if T carries disjoint almost limited weakly p-summable sequences of E to norm null sequences of X.

It is clear that alpc operators and disjoint p-convergent operators on a Banach lattice E are dalpc. However, $Id_{L^1[0,1]}$ is dalpc but it is not alpc. Also $Id_{C[0,1]}$ is dalpc but it is not disjoint p-convergent.

THEOREM 4.5. *For a Banach lattice E the following statements are equivalent:*

(a) E has the positive strong limited p-Schur property,

(b) $\mathscr{L}_{dalpc}(E,Y) = \mathscr{L}(E,Y)$, for each Banach space Y,

(c) $\mathscr{L}_{dalpc}(E,\ell_\infty) = \mathscr{L}(E,\ell_\infty)$.

Proof. It suffices to prove that $(c) \Rightarrow (a)$ Assume to the contrary that E does not have the positive strong limited p-Schur property. Then there exists a weakly p-summable and disjoint almost limited sequence (x_n) in E such that $\|x_n\| = 1$ for all n. Choose a normalized sequence (x_n^*) in E^* such that $|\langle x_n, x_n^* \rangle| = 1$ for all n. The operator $T : E \to \ell_\infty$ defined by $Tx = (\langle x, x_n^* \rangle)$, $x \in E$ is not dalpc. This leads to a contradiction. □

It is also a natural question to know which condition should be added so that the lpc operator is a dalpc operator and vice versa. The σ-Dedekind completeness is needed to answer this question and obtain the following characterization.

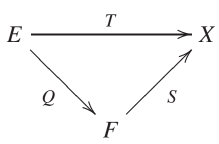
THEOREM 4.6. *Let X be a Banach space and E be a σ-Dedekind complete Banach lattice. Then for an operator $T : E \to X$, the following statements are equivalent:*

(a) T is an lpc operator,

(a) T is a dalpc operator,

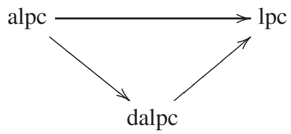
Proof. $(a) \Rightarrow (b)$ Let $T : E \to X$ be an lpc operator and E is σ-Dedekind complete. To show that T is dalpc, let (x_n) be a disjoint almost limited weakly p-summable sequence in E. Since E is a σ-Dedekind complete Banach lattice, it follows from [5, Theorem 2.8] that (x_n) is limited. Since T is lpc, $\|Tx_n\| \to 0$ which implies that T is dalpc.

$(b) \Rightarrow (a)$ First we show that T is order weakly compact. For this, let (x_n) be an arbitrary order bounded disjoint sequence in E. Then (x_n) is weakly p-suammble and almost limited, by the property (d). Therefore, $\|Tx_n\| \to 0$. This implies that T is order weakly compact, see [1, Theorem 5.57]. Therefore by [1, Theorem 5.58], T admits a factorization through a Banach lattice F with order continuous norm.



such that $Q : E \to F$ is a lattice homomorphism (cf. [1, Theorem 5.58]). Since each Banach lattice with order continuous norm has the limited p-Schur property, for each limited weakly p-summable sequence (x_n) in E, (Qx_n) is norm null and so $\|Tx_n\| = \|SQx_n\| \to 0$. This implies that T is lpc. □

Note that by [7], every weakly p-summable almost limited sequence in $C[0,1], c$ is norm null. That is, the identity operator on these two spaces is dalpc while it is not order weakly compact. From Theorem 4.6, in each Banach lattice with the property (d), we have the following diagram.



It is clear that each M-weakly compact operator $T : E \to X$ is dalpc. It follows from Theorem 4.6 and [34, Proposition 2.7] that each dalpc operator $T : E \to X$ from a Banach lattice E with the property (d) and the dual positive Schur property to a Banach space X, is M-weakly compact.

Let's examine where lpc operators, dalpc operators and alpc operators are the same? Recall that for each operator T from a Banach lattice E into a Banach space X, the lattice semi-norm q_T on E is defined by $q_T(x) = \sup_{|z| \leqslant |x|} \|Tz\|$ for all $x \in E$. Note that q_T is continuous for the norm of E. The reader can see the lattice semi-norm q_T on E in [27]. We can go quite further for σ-Dedekind complete almost limited weak p-consistent Banach lattices.

THEOREM 4.7. *Let $T : E \to F$ be an operator. Suppose that E is σ-Dedekind complete almost limited weak p-consistent. Then for all $p \geqslant 2$, the following assertions are equivalent:*

(a) T is a lpc operator,

(b) T is a dalpc operator,

(c) T is order weakly compact and each weakly p-compact almost limited subset of E is approximately order bounded with respect to the lattice seminorm q_T (which q_T is defined in [27]),

(d) T is an alpc operator.

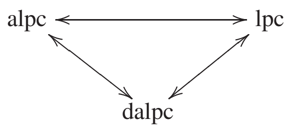
Proof. $(a) \Leftrightarrow (b)$ It follows from Theorem 4.6.

$(b) \Rightarrow (c)$ First note that by Theorem 4.6, T is order weakly compact. Now, let W be a weakly p-compact almost limited subset of E and $\varepsilon > 0$. We show that $q_T(x_n) \to 0$ for each disjoint sequence (x_n) in the solid hull A of W. For this, note that $q_T(x_n) = \sup\{\|Ty\| : |y| \leqslant |x_n|\}$ and so for each n there is y_n in A with $|y_n| \leqslant |x_n|$ and $q_T(x_n) \leqslant 2\|Ty_n\|$. It follows from Theorem 3.6 of [20] that (x_n) is weakly p-summable and also almost limited by property (d). On the other hand, the sequence (y_n) is a disjoint sequence in the solid hull of (x_n). From (a), $\|Ty_n\| \to 0$ which implies that $q_T(x_n) \to 0$. Thus by [1, Theorem 13.5], there exists some $u \in E^+$ in the order ideal generated by A, such that $q_T((|x| - u)^+)) \leqslant \varepsilon$, for all $x \in W$. Therefore W is approximately order bounded with respect to the lattice seminorm q_T.

$(c) \Rightarrow (d)$ Let (x_n) be an almost limited weakly p-summable sequence in E. Since each weakly p-compact almost limited subset of E is approximately order bounded with respect to q_T, there exists some $u \in E^+$ such that $\|T(x_n - u)^+\| < \frac{\varepsilon}{2}$ for all n. On the other hand, E is almost limited weak p-consistent, and so $(|x_n|)$ is an almost limited weakly p-summable sequence in E. Then we can assume that (x_n) is positive, replacing (x_n) with $(|x_n|)$ if necessary. Since $(x_n \wedge u)$ is an order bounded positive sequence in E and T is order weakly compact, by [27, Corollary 3.4.9], $\|T(x_n \wedge u)\| < \frac{\varepsilon}{2}$ for all n. Hence $\|T(x_n)\| \leqslant \|T(x_n - u)^+\| + \|T(x_n \wedge u)\| < \varepsilon$ for all n. This implies that $\|T(x_n)\| \to 0$ and so T is alpc.

$(d) \Rightarrow (a)$ It is clear. $\square$

From Theorem 4.7, in each almost limited weak p-consistent Banach lattice with the property (d), dalpc operators and alpc operators are the same. In each σ-Dedekind complete almost limited weak p-consistent Banach lattice, for all $p \geqslant 2$, we have the following diagram:



THEOREM 4.8. *For a Banach lattice E, the following statements are equivalent:*

(a) E has the positive strong limited p-Schur property.

(b) Every disjoint (positive) operator $T : E \to \ell_\infty$ is dalpc.

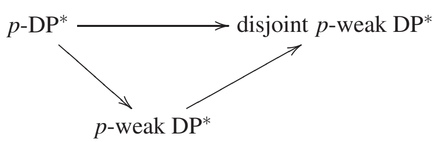
Proof. $(a) \Rightarrow (b)$ Obvious. In this case, each operator $T : E \to \ell_\infty$ is dalpc.

$(b) \Rightarrow (a)$ Let (x_n) be a disjoint almost limited weakly p-summable sequence in E, but $\|x_n\| = 1$ for all n. Then (x_n) is weakly null and so by [34, Proposition 1.3], $|x_n| \xrightarrow{w} 0$. Hence by [16, Corollary 2.6], there is bounded disjoint positive sequence (x_n^*) in E^* and an $\varepsilon > 0$ such that $x_n^*(x_n) \geqslant \varepsilon$, for all n. The disjoint (positive) operator $T : E \to \ell_\infty$ by $Tx = (\langle x, x_n^* \rangle)$, $x \in E$ is not dalpc and this leads to a contradiction. $\square$

The disjoint DP* property was defined by replacing disjoint weakly null sequences in the definition of the DP* property with weakly null sequences in [38]. The following definition arises naturally by doing the same with the definition of the p-weak DP* property introduced in [9].

DEFINITION 4.9. A Banach lattice E has the *disjoint p-weak DP* property*, if for every weakly p-summable disjoint sequence (x_n) in E and every disjoint weak* null sequence (f_n) in E^*, $f_n(x_n) \to 0$.

In other words, E has the disjoint p-weak DP* property if and only if each disjoint weakly p-summable sequence in E is almost limited. Each Banach lattice with the p-weak DP* property has the disjoint p-weak DP* property too. In fact, we have the following diagram:



THEOREM 4.10. *For a Banach lattice E, the following statements are equivalent:*

(a) E has the disjoint p-weak DP* property,

(b) Every disjoint operator $T : E \to c_0$ is disjoint p-convergent.

Proof. $(a) \Rightarrow (b)$ If E has the disjoint p-weak DP* property, then each disjoint weakly p-summable sequence in E is almost limited. From [21, Proposition 2.3], each disjoint operator $T : E \to c_0$ is alpc, and so it is dalpc. Hence for each each disjoint weakly p-summable sequence (x_n) in E, we have $\|Tx_n\| \to 0$ which implies that T is disjoint p-convergent.

$(b) \Rightarrow (a)$ Let $(x_n)_{n=1}^{\infty}$ be a disjoint weakly p-summable sequence in E, and let $(x_n^*)_{n=1}^{\infty}$ be a disjoint sequence in E^* satisfying $x_n^* \xrightarrow{w^*} 0$. By hypothesis, the disjoint operator $T : E \to c_0$ defined by $Tx = (x_n^*(x))$, $x \in E$ is disjoint p-convergent, and hence $|x_n^*(x_n)| \leqslant \|Tx_n\| \to 0$. This implies that E has the disjoint p-weak DP* property. □

The following theorem gives an operator characterization for the disjoint p-weak DP* property. It is clear that each disjoint p-convergent operator is dalpc.

THEOREM 4.11. *For a Banach lattice E the following assertions are equivalent.*

(a) E has the disjoint p-weak DP* property,

(b) $\mathscr{L}_{dalpc}(E,Y) = \mathscr{L}_{dpc}(E,Y)$, for each Banach space Y,

(c) $\mathscr{L}_{dalpc}(E,c_0) = \mathscr{L}_{dpc}(E,c_0)$.

Proof. $(a) \Rightarrow (b)$ Let $T \in \mathscr{L}_{dlpc}(E,X)$, and let $(x_n)_{n=1}^{\infty}$ be a disjoint weakly p-summable sequence in E. Since E has the disjoint p-weak DP* property, (x_n) is almost limited in E and by hypothesis, $\|T(x_n)\| \to 0$. This implies that T is disjoint p-convergent.

$(b) \Rightarrow (c)$ Obvious.

$(c) \Rightarrow (a)$ Let $(x_n)_{n=1}^{\infty}$ be a disjoint weakly p-summable sequence in E, and let $(x_n^*)_{n=1}^{\infty}$ be a disjoint sequence in E^* satisfying $x_n^* \xrightarrow{w^*} 0$. By [21], the operator $T : E \to c_0$ defined by $Tx = (x_n^*(x))$, $x \in E$ is alpc and so dalpc. Hence $|f_n(x_n)| \leqslant \|Tx_n\|$ holds for all $n \in \mathbb{N}$ and E has the disjoint p-weak DP* property. □

Some results concerning the p-positive Schur property are discussed in [38]. We conclude this section by stating another characterization for the p-positive Schur property, inspired by [9, Proposition 3.7]. It is under the p-weak DP* property that the strong limited p-Schur property coincide with the p-Schur property.

COROLLARY 4.12. *Let E be a Banach lattice. Then the following assertions are equivalent:*

(a) E has the p-positive Schur property,

(b) E has the positive strong limited p-Schur property and the disjoint p-weak DP* property.

Proof. $(a) \Rightarrow (b)$ Obvious.

$(b) \Rightarrow (a)$ It is enough to show that for disjoint positive weakly p-summable sequence (x_n) in E, $\|x_n\| \to 0$. To this end, note that since E has the disjoint p-weak DP* property, the sequence (x_n) is almost limited. On the other hand, E has the positive strong limited p-Schur property and so $\|x_n\| \to 0$. Thus E has the p-positive Schur property. □

Finally, note that although several classes of sets in the dual of Banach spaces, and especially in the dual of Banach lattices have been introduced and studied in [4, 23, 24, 25], but a class of strong L_p-limited sets in the dual of Banach lattices can have important results in functional analysis and operator theory. By introducing a class of strong L_p-limited sets in the dual of Banach lattices and their relationship to other well-known sets, several operator characterizations are considered. Also using disjoint sequences in Banach lattices, some classes of operators are defined that can be characterize the positive strong limited p-Schur property and disjoint p-weak DP* property. The relationship between some classes of operators such as p-convergent operators, almost limited p-convergent operators, limited p-convergent operators, disjoint limited p-convergent operators and unconditionally p-convergent operators are shown in some diagrams. Considering that every (V)-set in the dual of a Banach lattice is an L_p-limited set (Theorem 3.3), the relationship between the strong L_p-limited sets with different class of sets is also considered. Since in operator theory, the connection between different classes of known operators as well as known sets can lead to good results, these have been widely used in this article. To make it easier for the reader to find the concepts used in the article, we have created the following table at the end of this article.

A summary table of key definitions

Since the numerous definitions and abbreviations make the text dense and sometimes hard to follow, a summary table of key definitions would enhance readability. We have created the following table that may be helpful to readers.

Name	Definition	Abbreviation
limited set	$C \subset X$ norm bounded; every weak*-null sequence in X^* converges uniformly to 0 on C.	–
Dunford-Pettis set	Same as above, but with the weakly null sequence.	DP

almost limited set	$C \subset E$ norm bounded; every disjoint weak*-null sequence in E^* converges uniformly to 0 on C.	–
almost DP set	Same as above, but with the disjoint weakly null sequence.	–
L-limited set of order p	$A \subset X^*$; for every limited weakly p-summable sequence (x_n) in X, $\sup_{f\in A} \lvert f(x_n)\rvert \to 0$.	L_p-limited set
strong L-limited set of order p	$A \subset E^*$; for every almost limited weakly p-summable sequence (x_n) in E, $\sup_{f\in A} \lvert f(x_n)\rvert \to 0$.	strong L_p-limited set
(V)-set	$A \subset X^*$; for every every w.u.c. series $\sum x_n$ in X, $\sup_{f\in A} \lvert f(x_n)\rvert \to 0$.	(V)-set
p-(V)-set or (weakly p-L-set)	For every weakly p-summable sequence in X converges uniformly to 0 on A.	p-(V)-set
Gelfand-Phillips property	Every limited set in X is relatively compact.	GP property
relatively compact DP property	Every DP set in X is relatively compact.	$\mathrm{DP}_{rc}\mathrm{P}$
DP* property	Every relatively weakly compact set in X is limited.	DP* property
DP property	Every relatively weakly compact set in X is DP.	–
Schur property	Every weakly null sequence in X is norm null.	–
Grothendieck property	Every weak*-null sequence in X^* is weakly null.	–
strong GP property	Every almost limited set in E is relatively compact.	–
strong $\mathrm{DP}_{rc}\mathrm{P}$	Every almost DP set in E is relatively compact.	–

weak DP* property	Every relatively weakly compact set in E is almost limited.	–
weak DP property	Every relatively weakly compact set in E is almost DP.	–
DP* property of order p	Every relatively weakly p-compact set in X is limited.	p-DP* property
Schur property of order p	Every weakly p-summable sequence in X is norm null.	p-Schur property
limited p-Schur property or GP property of order p	Every limited weakly p-summable sequence in X is norm null.	p-GP property
weak DP* property of order p	Every weakly p-summable sequence in E is almost limited.	p-weak DP* property
positive Schur property of order p	Every weakly p-summable sequence in E^+ is norm null.	p-positive Schur property
strong limited p-Schur property	Every almost limited weakly p-summable sequence in E is norm null.	–
positive strong limited p-Schur property	Every disjoint almost limited weakly p-summable sequence in E is norm null.	–
disjoint p-weak DP* property	For every disjoint weakly p-summable sequence (x_n) in E and every disjoint weak*-null sequence (f_n) in E^*, one has $f_n(x_n) \to 0$.	–
order weakly compact	$\|T(x_n)\| \to 0$ for every order bounded disjoint sequence $(x_n) \subset E$.	–
unconditionally convergent	$\|T(x_n)\| \to 0$ for every weakly 1-summable sequence $(x_n) \subset X$ (or every w.u.c. series $\sum x_n$ in X).	uc
p-convergent	$\|T(x_n)\| \to 0$ for every weakly p-summable sequence $(x_n) \subset X$.	pc

disjoint almost limited p-convergent operator	$\|T(x_n)\| \to 0$ for every disjoint almost limited weakly p-summable sequence $(x_n) \subset E$.	dalpc
limited p-convergent operator	$\|T(x_n)\| \to 0$ for every limited weakly p-summable sequence $(x_n) \subset X$.	lpc

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