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SOME RESULTS ABOUT A QUASILINEAR SINGULAR PARABOLIC EQUATION

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Abstract. We investigate the following quasilinear parabolic and singular equation,

$$\begin{cases} u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(x,u) \text{ in } (0,T)\times\Omega, \\ u = 0 \text{ on } (0,T)\times\partial\Omega, \quad u > 0 \text{ in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega, \end{cases} \tag{P$_t$}$$

where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^N$, $1 < p < \infty$, $0 < \delta$ and $T > 0$. We assume that $(x,s) \in \Omega\times\mathbb{R}^+ \to f(x,s)$ is a bounded below Caratheodory function, asymptotically sub-homogeneous, i.e.

$$\begin{cases} \text{if } p \leqslant 2,\ 0 \leqslant \limsup\limits_{t\to+\infty} \dfrac{f(x,t)}{t^{p-1}} = \alpha_f < \lambda_1(\Omega), \\ \text{if } p > 2,\ 0 \leqslant \limsup\limits_{t\to+\infty} \dfrac{f(x,t)}{t} = \alpha_f < \infty, \end{cases} \tag{0.1}$$

(where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_p$ in Ω with homogeneous Dirichlet boundary conditions) and $u_0 \in W_0^{1,p}(\Omega)$. Then, for any $\delta \in (0,1)$, we prove for any $T > 0$ the existence of a weak solution $u \in \mathbf{V}(Q_T)$ to $(\mathrm{P_t})$. The proof involves a semi-discretization in time approach and the study of the stationary problem associated to $(\mathrm{P_t})$. The key points in the proof is to show that the approximated solutions remain (uniformly) positive in any compact K of Ω and from energy estimates converges to a weak solution to $(\mathrm{P_t})$. Next, under additional assumptions on the initial data, δ and the nonlinearity f, we prove long time convergence of global weak solutions in $W_0^{1,p}(\Omega)$. This stabilization property is established by proving an additional energy estimate and by using the regularity result in Simon [23]. These results extend with a different approach a previous work of the authors ([3]) regarding the problem $(\mathrm{P_t})$ where existence and uniqueness of solutions are proved under a cone condition on the initial data and via the theory of nonlinear accretive operators.

Mathematics subject classification (2010): 35J65, 35J20, 35J70.
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REFERENCES

[1] C. ARANDA AND T. GODOY, *Existence and multiplicity of positive solutions for a singular problem associated to the p-Laplacian operator*, Electron. J. Differential Equations, **132** (2004), 1–15.
[2] R. ARIS, *Mathematical modelling techniques*, Vol. **24** of Research Notes in Mathematics, Pitman (Advanced Publishing Program), Boston, Mass., 1979.
[3] M. BADRA, K. BAL AND J. GIACOMONI, *A singular parabolic equation: existence, stabilization*, to appear.
[4] H.T. BANKS, *Modeling and control in the biomedical sciences*, Vol. **6** Lecture Notes in Biomathematics, Springer-Verlag, Berlin, 1975.

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[5] L. BOCCARDO AND L. ORSINA, *Semilinear elliptic equations with singular nonlinearities*, Calc. Var. Partial Differential Equations, **37**, (3-4) (2010), 363–380.

[6] H. BREZIS, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.

[7] M.G. CRANDALL, P.H. RABINOWITZ AND L. TARTAR, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations, **2**, (2) (1977), 193–222.

[8] J. DÁVILA AND M. MONTENEGRO, *Existence and asymptotic behavior for a singular parabolic equation*, Trans. Amer. Soc., **357** (5) (2005), 1801–1828.

[9] J.I. DÍAZ, *Nonlinear partial differential equations in free boundaries. Vol. I*, volume **106** of Research Notes in Mathematics, Pitman (Advanced Publishing Program), Boston, M.A., 1985.

[10] J.I. DÍAZ AND J. E. SAÁ *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C.R. Acad. Sci. Paris Sér. I. Math., **305** (12) (1987), 521–524.

[11] M. GHERGU AND V.D. RADULESCU, *Multi-parameter bifurcation and asymptotics behavior for the singular Lane-Emden-Fowler equation with a convection term*, Proc. Roy. Soc. Edinburgh Sect. A, **135**, (1) (2005), 61–83.

[12] M. GHERGU AND V.D. RADULESCU, *Singular elliptic problems: bifurcation and asymptotic analysis*, volume **37** of Oxford Lecture Series in Mathematics and its Applications. The Clarendon Press Oxford University Press, Oxford, 2008.

[13] J. GIACOMONI, I. SCHINDLER AND P. TAKÁČ, *Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation*, Ann. Sc. Norm. Super. Pisa Cl Sci. (5), **6**, (1) (2007), 117–158.

[14] J. HERNÁNDEZ, F. MANCEBO, *Singular elliptic and parabolic equations*, Handbook of Differential Equations, **3** (2006), 317–400.

[15] J. HERNÁNDEZ, F. MANCEBO AND J.M. VEGA, *On the linearization of some singular, nonlinear elliptic problems and applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **19**, (6) (2002), 777–813.

[16] H.B. KELLER AND D.S. COHEN, *Some positone problems suggested by nonlinear heat generation*, J. Math. Mech., **16** (1967), 1361–1376.

[17] C.D. LUNING AND W.L. PERRY, *Positive solutions of negative exponent generalized Emden-Fowler boundary value problems*, SIAM J. Math. Anal., **12**, (6) (1981), 874–879.

[18] A. NACHMAN AND A. CALLEGARI, *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids*, SIAM J. Appl. Math., **38**, (2) (1980), 275–281.

[19] D. O'REGAN, *Some general existence principles and results for* $(\phi(y'))' = qf(t,y,y'),\quad 0<t<1$, SIAM J. Math. Anal., **24**, (3) (1993), 648–668.

[20] K. PERERA AND E.A.B. SILVA, *On singular* p*-Laplacian problems*, Differential Integral Equations, **20**, (1) (2007), 105–120.

[21] W.L. PERRY, *A monotone iterative technique for solution of* p*-order (* $p<0$ *) reaction-diffusion problems in permeable catalysis*, J. Comput. Chem., **5**, (4) (1984), 353–357.

[22] J. SIMON, *Régularité de la solution d'un problème aux limites nonlinéaires*, Ann. Fac. Sci. Toulouse Math. (5), **3**, (3-4) (1981), 247–274.

[23] J. SIMON, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4), **146** (1987), 65–96.

[24] P. TAKÁČ, *Stabilization of positive solutions for analytic gradient-like systems*, Discrete Contin. Dynam. Systems, **6**, (4) (2000), 947–973.

[25] H. TRIEBEL, *Interpolation theory, functions spaces, differential operators*, Johann Ambrosius Barth, Heidelberg, second edition, 1995.

[26] M. WINKLER, *Nonuniqueness in the quenching problem*, Math. Ann., **339** (3) (2007), 559–597.

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