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A SECOND–ORDER DIFFERENTIAL SYSTEM WITH HESSIAN–DRIVEN DAMPING; APPLICATION TO NON–ELASTIC SHOCK LAWS

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Abstract. We consider the second-order differential system with Hessian-driven damping $\ddot{u}+\alpha\dot{u}+\beta\nabla^2\Phi(u)\dot{u}+\nabla\Phi(u)+\nabla\Psi(u)=0$, where $\mathscr{H}$ is a real Hilbert space, $\Phi,\Psi:\mathscr{H}\to\mathbb{R}$ are scalar potentials, and α,β are positive parameters. An interesting property of this system is that, after introduction of an auxiliary variable y, it can be equivalently written as a first-order system involving only the time derivatives $\dot{u}$, $\dot{y}$ and the gradient operators $\nabla\Phi$, $\nabla\Psi$. This allows to extend our analysis to the case of a convex lower semicontinuous function $\Phi:\mathscr{H}\to\mathbb{R}\cup\{+\infty\}$, and so to introduce constraints in our model. When $\Phi=\delta_K$ is the indicator function of a closed convex set $K\subseteq\mathscr{H}$, the subdifferential operator $\partial\Phi$ takes account of the contact forces, while $\nabla\Psi$ takes account of the driving forces. In this setting, by playing with the geometrical damping parameter β, we can describe nonelastic shock laws with restitution coefficient. Taking advantage of the infinite dimensional framework, we introduce a nonlinear hyperbolic PDE describing a damped oscillating system with obstacle. The first-order system is dissipative; each trajectory weakly converges to a minimizer of $\Phi+\Psi$, provided that Φ and $\Phi+\Psi$ are convex functions. Exponential stabilization is obtained under strong convexity assumptions.

Mathematics subject classification (2010): 34C35, 34D05, 65C25, 90C25, 90C30.

Keywords and phrases: asymptotic stabilization, convex variational analysis, dissipative dynamical systems, exponential stabilization, gradient-like systems, Hessian-driven damping, impact dynamics, nonelastic shocks, nonsmooth potentials, restitution coefficient, second-order nonlinear differential equations, unilateral mechanics, viscoelastic membrane.

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