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EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO A $p(x)$–LAPLACIAN EQUATION WITH NONLINEAR BOUNDARY CONDITION ON UNBOUNDED DOMAIN

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Abstract. We study the existence and multiplicity of positive solutions for the nonlinear boundary value problems involving the $p(x)$-Laplacian of the form

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u)+b(x)|u|^{p(x)-2}u=f(x,u) & \text{in } \Omega\subset\mathbb{R}^N,\\ a(x)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}=g(x,u) & \text{on } \Gamma=\partial\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is an unbounded domain with non-compact, smooth boundary $\Gamma=\partial\Omega$, $p\in C^{0,1}(\Omega)$ and $1<p^-\leqslant p(x)\leqslant p^+<N$, a,b are suitable weights. By using the variational methods, we prove that there exist multiple solutions provided f and g are given appropriate assumptions.

Mathematics subject classification (2010): 35J35,35J40,35J67,35J70.
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