

## ON THE UNIQUENESS OF SOLUTIONS OF TWO INVERSE PROBLEMS FOR THE SUBDIFFUSION EQUATION

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**Abstract.** Consider an arbitrary positive self-adjoint operator  $A$  defined in a separable Hilbert space  $H$ . In the nonlocal boundary value problem  $D_t^\rho u(t) + Au(t) = f(t)$  ( $0 < \rho < 1$ ,  $0 < t \leq T$ ),  $u(\xi_0) = \alpha u(0) + \varphi$  ( $\alpha$  is a constant and  $0 < \xi_0 \leq T$ ), where  $D_t$  is the Caputo derivative, assume that the right-hand side of the equation or the function  $\varphi$  is unknown. In this paper, we study the inverse problems of determining these unknown functions. For both inverse problems,  $u(\xi_1) = V$  is taken as the over-determination condition. The main attention is paid to the study of the influence of the constant  $\alpha$  on the existence and uniqueness of the solution to the problems. An interesting effect was discovered: when solving the forward problem, the uniqueness of the solution  $u(t)$  was violated, while when solving the inverse problem for the same values of  $\alpha$ , the solution  $u(t)$  became unique.

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