

## SOME REFINEMENTS OF HERMITE–HADAMARD INEQUALITY USING $k$ –FRACTIONAL CAPUTO DERIVATIVES

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*Abstract.* In this work using  $k$ -fractional Caputo derivatives we obtain some versions of the Hadamard inequality for the function  $f$  such that  $f(n)$  is  $(h, m)$ -convex modified of the second type. Throughout the work, we show that some known results from the literature can be obtained as particular cases of the results presented here.

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