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A SERIES REPRESENTATION FOR RIEMANN'S ZETA FUNCTION AND SOME INTERESTING IDENTITIES THAT FOLLOW

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Abstract. Using Cauchy's Integral Theorem as a basis, what may be a new series representation for Dirichlet's function $\eta(s)$, and hence Riemann's function $\zeta(s)$, is obtained in terms of the Exponential Integral function $E_s(i\kappa)$ of complex argument. From this basis, infinite sums are evaluated, unusual integrals are reduced to known functions and interesting identities are unearthed. The incomplete functions $\zeta^{\pm}(s)$ and $\eta^{\pm}(s)$ are defined and shown to be intimately related to some of these interesting integrals. An identity relating Euler, Bernoulli and Harmonic numbers is developed. It is demonstrated that a known simple integral with complex endpoints can be utilized to evaluate a large number of different integrals, by choosing varying paths between the endpoints.

Mathematics subject classification (2010): 11M06, 11M26, 11M35, 11M99, 26A09, 30B40, 30E20, 33C20, 33B20, 33B99.

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Paper JCA-17-09

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