

A NATURAL COMPANION TO CATALAN'S CONSTANT

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Abstract. We consider here a trilogarithmic expression that plays a similar role to Catalan's constant G in many ways:

$$\mathcal{G} := \Im \left(\operatorname{Li}_3 \left(\frac{i+1}{2} \right) \right).$$

The main purpose of this article is to demonstrate how \mathcal{G} is a naturally occurring and useful expression that deserves to be recognized as a mathematical constant and as a natural trilogarithmic "extension" of Catalan's constant G . Having identified this constant, we evaluate many new and non-trivial integrals, Euler-type sums, ${}_pF_q$ series, and binomial-harmonic series using \mathcal{G} , extending known results on the classical version of Catalan's constant.

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