

## SHARPENING OF BERNSTEIN AND TURÁN-TYPE INEQUALITIES FOR POLYNOMIALS

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*Abstract.* Let  $p(z)$  be a polynomial of degree  $n$ . The polar derivative of  $p(z)$  with respect to a real or complex number  $\alpha$  is defined by

$$D_{\alpha}p(z) = np(z) + (\alpha - z)p'(z).$$

Govil and Mctume [Acta Math. Hungar., 104, 115–126 (2004)] proved that if  $p(z)$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ ,  $k \geq 1$ , then for any complex number  $\alpha$  with  $|\alpha| \geq 1 + k + k^n$ ,

$$\max_{|z|=1} |D_{\alpha}p(z)| \geq n \left( \frac{|\alpha| - k}{1 + k^n} \right) \max_{|z|=1} |p(z)| + n \left( \frac{|\alpha| - (1 + k + k^n)}{1 + k^n} \right) \min_{|z|=k} |p(z)|.$$

In this paper, we prove an improvement of the above inequality. Further, we prove an improvement of a result due to Govil [Proc. Natl. Acad. Sci., 50, 50–52 (1980)].

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