Journal of Classical Analysis
Volume 19, Number 2 (2022), 139–147 doi:10.7153/jca-2022-19-11

AN ELEMENTARY PROOF OF RAMANUJAN'S IDENTITY FOR ODD ZETA VALUES

SARTH CHAVAN

Abstract. The main purpose of this article is to present an elementary proof of Ramanujan's identity for odd zeta values. Our proof solely relies on a Mittag-Leffler type expansion for hyperbolic cotangent function and Euler's identity for even zeta values.

Mathematics subject classification (2020): Primary: 11M06.
Keywords and phrases: Riemann zeta function, Ramanujan's identity for odd zeta values.

REFERENCES

[1] G. E. ANDREWS AND B. C. BERNDT, *Ramanujan's Lost Notebook*, Part IV, Springer, New York, 2013.

[2] R. APÉRY, *Irrationalité de* $\zeta(2)$ *et* $\zeta(3)$, Astérisque **61** (1979), 11–13.

[3] T. M. APOSTOL, *Another elementary proof of Euler's formula for* $\zeta(2n)$, Amer. Math. Monthly, 425–431, 1973.

[4] K. BALL AND T. RIVOAL, *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs* (French), Invent. Math. **146** no. 1 (2001), 193–207.

[5] B. C. BERNDT, *Ramanujan's notebooks*, Part II, Springer, New York, 1989.

[6] B. C. BERNDT AND A. STRAUB, *Ramanujan's formula for* $\zeta(2n+1)$, *Exploring the Riemann Zeta Function*, H. Montgomeryand, A. Nikeghbali and M. Rassias, Eds., Springer, Cham, 2017, 13–34.

[7] B. C. BERNDT, *Ramanujan's Notebooks*, Part III, Springer, New York, 1991.

[8] B. C. BERNDT, *Modular transformations and generalizations of several formulae of Ramanujan*, Rocky Mountain J. Math. **7** (1977), 147–189.

[9] B. C. BERNDT AND A. STRAUB, *On a secant Dirichlet series and Eichler integrals of Eisenstein series*, Math. Z. 284 no. 3–4 (2016), 827–852.

[10] F. BEUKERS, *A note on the irrationality of* $\zeta(2)$ *and* $\zeta(3)$, Bull. London Math. Soc. 11:3 (1979), 268–272.

[11] F. BROWN, *Irrationality proofs for zeta values, moduli spaces and dinner parties*, Mosc. J. Comb. Number Theory, **6** no. 2–3 (2016), 102–165.

[12] S. FISCHLER, *Irrationalité de valeurs de ẑeta (d'aprés Apéry, Rivoal, ...)* (Séminaire Bourbaki 2002-2003, exposé numéro 910, 17 November 2002) Astérisque **294** (2004), 27–62.

[13] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series and Products*, Edited by D. Zwillinger and V. H. Moll, Eighth Edition, Academic Press, 2014.

[14] E. GROSSWALD, *Comments on some formulae of Ramanujan*, Acta Arith. **21** (1972), 25–34.

[15] E. GROSSWALD, *Die Werte der Riemannschen Zetafunktion an ungeraden Argumentstellen*, Nachr. Akad. Wiss. Göttinger Math.-Phys. Kl. II (1970), 9–13.

[16] S. GUN, M. R. MURTY AND P. RATH, *Transcendental values of certain Eichler integrals*, Bull. London Math. Soc., **43** no. 5 (2011), 939–952.

[17] A. K. HAYNES AND W. ZUDILIN, *Hankel determinants of zeta values*, SIGMA Symmetry Integrability Geom. Methods Appl. **11**, 101, (2015), 1–5.

[18] P. KIRSCHENHOFER AND H. PRODINGER, *On some applications of formulae of Ramanujan in the analysis of algorithms*, Mathematika **38** no. 1 (1991), 14–33.

[19] M. LERCH, *Sur la fonction* $\zeta(s)$ *pour valeurs impaires de l'argument*, J. Sci. Math. Astron. pub. pelo Dr. F. Gomes Teixeira, Coimbra **14** (1901), 65–69.

Paper JCA-19-11

[20] S. L. MALURKAR, *On the application of Herr Mellin's integrals to some series*, J. Indian Math. Soc. **16** (1925/26), 130–138.

[21] T. RIVOAL, *La fonction ẑeta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris Sér. I Math. 331:4 (2000), 267–270.

[22] T. RIVOAL, W. ZUDILIN, *A note on odd zeta values*, Sém. Lothar. Combin. **81** (2020), Art. B81b.

[23] H. F. SANDHAM, *Some infinite series*, Proc. Amer. Math. Soc. **5** (1954), 430–436.

[24] F. P. SAYER, *The sums of certain series containing hyperbolic functions*, Fibonacci Quart, **14**, 1976.

[25] J. R. SMART, *On the values of the Epstein zeta function*, Glasgow Math. J. **14** (1973), 1–12.

[26] G. N. WATSON, *Theorems stated by Ramanujan (ii): Theorems on summation of series*, J. London Math. Soc. **3** (1928), 216–225.

[27] W. ZUDILIN, *One of the numbers* $\zeta(5),\ \zeta(7),\ \zeta(9),\ \zeta(11)$ *is irrational*, Uspekhi Math. Nauk 56:4 (2001).