Journal of
Classical
Analysis
Volume 20, Number 2 (2022), 79–101
doi:10.7153/jca-2022-20-07

EXPLICIT EXPRESSIONS FOR SOME LINEAR EULER–TYPE SUMS CONTAINING HARMONIC AND SKEW–HARMONIC NUMBERS

SEÁN M. STEWART

Abstract. Closed-form expressions for the three general linear Euler-type sums containing the harmonic numbers H_n and the skew-harmonic numbers $\overline{H}_n$

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(2n+1)^{2q+1}}, \sum_{n=1}^{\infty} \frac{\overline{H}_n}{(2n+1)^q}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n}{(2n+1)^{2q+1}},$$

where q is a positive integer greater than or equal to zero or two as needed to ensure convergence are given. Closed-form expressions for several other closely related generalised logarithmic integrals and sums are also presented.

Mathematics subject classification (2020): 11M06, 65B10.
Keywords and phrases: Euler sum, harmonic number, skew-harmonic number, Dirichlet beta function, polygamma function, logarithmic integrals, closed-form.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series 55 (10th printing), National Bureau of Standards, Washington, DC, 1972.

[2] G. BASTIEN, *Elementary methods for evaluating Jordan's sums* $\sum_{n\geqslant 1}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)\frac{1}{n^{2a}}$ *and* $\sum_{n\geqslant 1}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)\frac{1}{(2n-1)^{2a}}$ *and analogous Euler's type sums and for setting a* σ*-sum theorem*, arXiv:1301.7662 (2013).

[3] N. BHANDARI, *Problem 5659*, Sch. Sci. Math. **121**, 7 (2021), 448.

[4] G. BOROS AND V. MOLL, *Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals*, Cambridge University Press, Cambridge, 2004.

[5] K. N. BOYADZHIEV, *Power series with skew-harmonic numbers, dilogarithms, and double integrals*, Tatra Mt. Math. Publ. **56**, 1 (2013), 93–108.

[6] K. N. BOYADZHIEV, *Special Techniques for Solving Integrals: Examples and Problems*, World Scientific, New Jersey, 2022.

[7] J. M. CAMPBELL AND A. SOFO, *An integral transform related to series involving alternating harmonic numbers*, Integral Transforms Spec. Funct. **28**, 7 (2017), 547–559.

[8] J. M. CAMPBELL, P. LEVRIE, AND A. SOFO, *A natural companion to Catalan's constant*, J. Class. Anal. **18**, 2 (2021), 117–135.

[9] L. EULER, *Meditationes circa singulare sererum genus*, Novi Comm. Acad. Sci. Petropol. **20** (1776), 140–186. Reprinted in: *Opera Omnia ser. I*, vol. 15, B. G. Teubner, Leipzig, 1927, pp. 217–267.

[10] P. FLAJOLET AND B. SALVY, *Euler sums and contour integral representations*, Exp. Math. **7**, 1 (1998), 15–35.

[11] C. GEORGHIOU AND A. N. PHILIPPOU, *Harmonic sums and the zeta function*, Fibonacci Quart. **21**, 1 (1983), 29–36.

[12] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series and Products* (8th ed.), Academic Press, San Diego, 2015.

[13] K. S. KÖLBIG, *The polygamma function* $\psi^{(k)}(x)$ *for* $x=\frac{1}{4}$ *and* $x=\frac{3}{4}$, J. Comput. Appl. Math. **75**, 1 (1996), 43–46.

[14] L. LEWIN, *Polylogarithms and Associated Functions*, North Holland, New York, 1981.
[15] M. MATTMÜLLER AND F. LEMMERMEYER (Eds.), *Correspondence of Leonhard Euler with Christian Goldbach* (Vol. 1), Springer, Basel, 2015.
[16] N. NIELSEN, *Handbuch der Theorie der Gammafunktion*, B. G. Teubner, Leipzig, 1906.
[17] A. S. OLAIKHAN, *An Introduction to the Harmonic Series and Logarithmic Integrals: For High Schools Students up to Researchers*, Ali Shadhar Olaikhan, Phoenix, AZ, 2021.
[18] K. B. OLDHAM, J. C. MYLAND, AND J. SPANIER, *An Atlas of Functions: With Equator, the Atlas Function Calculator* (2nd ed.), Springer, New York, 2009.
[19] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT, AND C. W. CLARK, *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
[20] R. SITA RAMA CHANDRA AND A. SIVA RAMA SARMA, *Some identities involving the Riemann zeta function*, Indian J. Pure Appl. Math. **10**, 5 (1979), 602–607.
[21] R. SITARAMACHANDRARAO, *A formula of S. Ramanujan*, J. Number Theory **25**, 1 (1987), 1–19.
[22] A. SOFO AND D. CVIJOVIĆ, *Extensions of Euler harmonic sums*, Appl. Anal. Discrete Math. **6**, 2 (2012), 317–328.
[23] A. SOFO AND A. S. NIMBRAN, *Euler sums and integral connections*, Mathematics **7**, 9 (2019), Article 833, 24 pp.
[24] A. SOFO, *A family of definite integrals*, Scientia. Series A: Mathematical Sciences **31** (2021), 61–74.
[25] A. SOFO, *Log-hyperbolic tangent integrals and Euler sums*, Euro-Tbilisi Math. J. **15**, 2 (2022), 13–27.
[26] S. M. STEWART, *How to Integrate It: A Practical Guide to Finding Elementary Integrals*, Cambridge University Press, Cambridge, 2018.
[27] S. M. STEWART, *Explicit evaluation of some quadratic Euler-type sums containing double-index harmonic numbers*, Tatra Mt. Math. Publ. **77**, 1 (2020), 73–98.
[28] S. M. STEWART, *In the shadow of Euler's greatness: Adventures in the rediscovery of an intriguing sum*, Math. Intelligencer **43**, 3 (2021), 82–91.
[29] S. M. STEWART, *Using exponential generating functions to evaluate definite integrals*, Scientia. Series A: Mathematical Sciences **32** (2022), 99–133.
[30] G. T. WILLIAMS, *A new method of evaluating* $\zeta(2n)$, Amer. Math. Monthly **60**, 1 (1953), 19–25.
[31] C. XU, M. ZHANG, AND W. ZHU, *Some evaluation of harmonic number sums*, Integral Transforms Spec. Funct. **27**, 12 (2016), 937–955.