

SERIES INVOLVING POLYGAMMA FUNCTIONS  
AND CERTAIN VARIANT EULER HARMONIC SUMS

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*Abstract.* In this paper, we begin by introducing integral formulas related to the psi functions. Following that, we delve into specific series that involve polygamma functions, leveraging Eulerian numbers to express them as finite series of double integrals. Subsequently, we employ the findings from the second part to study variant Euler harmonic sums. Finally, we offer closed-form evaluations for a number of distinct instances of these variant Euler harmonic sums.

*Mathematics subject classification (2020):* 11M06, 11M35, 26B15, 33B15, 42A70, 65B10.

*Keywords and phrases:* Euler harmonic sums, variant Euler harmonic sums, polygamma functions, harmonic numbers, extended harmonic numbers, mathematical constants.

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