

D’ALEMBERT’S OTHER FUNCTIONAL EQUATION WITH AN AUTOMORPHISM

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Abstract. Let S be a semigroup, and $\sigma : S \rightarrow S$ an automorphism that need not be involutive. We determine the complex-valued solutions of the following functional equation

$$f(xy) - \mu(y)f(\sigma(y)x) = 2f(x)g(y), \quad x, y \in S,$$

where $\mu : S \rightarrow \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. This enables us to solve the functional equation

$$f(x\varphi(y)) - f(\psi(y)x) = 2f(x)g(y), \quad x, y \in S,$$

where $\varphi, \psi : S \rightarrow S$ are automorphisms such that φ is involutive and ψ is not necessarily involutive. Some consequences of these results are presented.

Mathematics subject classification (2020): 39B52, 39B32.

Keywords and phrases: Semigroup, d’Alembert’s equation, automorphism, multiplicative function.

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