

AN EXTENSION OF GLASSER'S MASTER THEOREM AND A COLLECTION OF IMPROPER INTEGRALS MANY OF WHICH INVOLVE RIEMANN'S ZETA FUNCTION

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Abstract. Glasser's Master Theorem [13] is essentially a restatement of Cauchy's integral Theorem reduced to a specialized form. Here we extend that theorem by introducing two new parameters, but still retain a simple form. Because of wide interest in entities involving Riemann's zeta function, the focus is on the evaluation of improper integrals with almost arbitrary integrands involving that function, but we also consider some other instances, the purpose being to demonstrate the power of the extended theorem. This is achieved by the presentation of a large number of examples that illustrate the ubiquity of the range of possibilities. One simple outcome of the study is the use of the extended theorem to show how it is possible to evaluate an integral when series or other representations of an integrand function do not converge.

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