

COMPLETENESS OF THE SYSTEMS OF BESSEL  
FUNCTIONS OF NEGATIVE HALF-INTEGER  
INDEX LESS THAN  $-1$  IN WEIGHTED  $L^2$ -SPACES

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**Abstract.** We establish necessary and sufficient conditions for the completeness of the system  $\{\rho_k^m \sqrt{x\rho_k} J_{-m-1/2}(x\rho_k) : k \in \mathbb{N}\}$  in the space  $L^2((0;1);x^{2m}dx)$  in terms of sequences of zeros of functions from certain classes of entire functions, where  $L^2((0;1);t^{2m}dt)$  be the weighted Lebesgue space of all measurable functions  $f : (0;1) \rightarrow \mathbb{C}$ , satisfying  $\int_0^1 t^{2m}|f(t)|^2 dt < +\infty$ ,  $m \in \mathbb{N}$ ,  $J_{-m-1/2}$  be the Bessel function of the first kind of index  $-m-1/2$  and  $(\rho_k)_{k \in \mathbb{N}}$  be a sequence of distinct nonzero complex numbers. We also obtain an analog of the Paley-Wiener theorem related to this system.

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