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GRAND FURUTA INEQUALITY AND ITS VARIANT

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Abstract. The grand Furuta inequality (GFI) is understood as follows: If positive operators A and B on a Hilbert space satisfy $A \geqslant B \geqslant 0$, A is invertible and $t \in [0,1]$, then

$$A^{1-t+r} \geqslant (A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}})^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $p,\ s \geqslant 1$ and $r \geqslant t$. In this note, we present a short proof of (GFI) which is done by the usual induction on s and the use of the Furuta inequality. Furthermore we propose another simultaneous extension of the Ando-Hiai and Furuta inequalities: If $A \geqslant B \geqslant 0$, A is invertible and $t \in [0,1]$, then

$$A^t \sharp_{\frac{1-t}{p-t}} B^p \geqslant A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p)$$

holds for $r \geqslant t$ and $p,\ s \geqslant 1$. Here $\sharp_\alpha$ is the α-geometric mean and $\natural_s$ for $s \notin [0,1]$ is of the same form as $\sharp_\alpha$.

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REFERENCES

[1] T. ANDO AND F. HIAI, *Log majorization and complementary Golden-Thompson type inequality*, Linear Alg. Appl., **197** (1994), 113–131.
[2] M. FUJII, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory, **23** (1990), 67–72.
[3] M. FUJII AND E. KAMEI, Ando-Hiai inequality and Furuta inequality, Linear Alg. Appl., **416** (2006), 541–545.
[4] M. FUJII, T. FURUTA AND E. KAMEI, *Furuta's inequality and its application to Ando's theorem*, Linear Alg. Appl., **179** (1993), 161–169.
[5] M. FUJII, E. KAMEI AND R. NAKAMOTO, *An analysis on the internal structure of the celebrated Furuta inequality*, Sci. Math. Japon., **62** (2005), 421–427.
[6] T. FURUTA, *$A \geqslant B \geqslant 0$ assures $(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0, p \geqslant 0, q \geqslant 1$ with $(1+2r)q \geqslant p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85–88.
[7] T. FURUTA, *Elementary proof of an order preserving inequality*, Proc. Japan Acad., **65** (1989), 126.
[8] T. FURUTA, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Alg. Appl., **219** (1995), 139–155.
[9] T. FURUTA, *Invitation to Linear Operators*, Taylor & Francis, London and New York, (2001).
[10] F. HIAI, *Log-majorizations and norm inequalities for exponential operators*, Linear Operators Banach Center Publications, vol. 38, 1997.
[11] E. KAMEI, *A satellite to Furuta's inequality*, Math. Japon., **33** (1988), 883–886.
[12] E. KAMEI, *Parametrization of the Furuta inequality*, Math. Japon., **49** (1999), 65–71.
[13] E. KAMEI, *Parameterized grand Furuta inequality*, Math. Japon., **50** (1999), 79–83.
[14] F. KUBO AND T. ANDO, *Means of positive linear operators*, Math. Ann., **246** (1980), 205–224.

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