Journal of
Mathematical
Inequalities
Volume 8, Number 1 (2014), 1–54 doi:10.7153/jmi-08-01

HOW TO SOLVE THREE FUNDAMENTAL LINEAR MATRIX INEQUALITIES IN THE LÖWNER PARTIAL ORDERING

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Abstract. This paper shows how to derive analytical solutions of the three fundamental linear matrix inequalities

$$
\begin{aligned}
&AXB \succcurlyeq C\,(\succ C, \preccurlyeq C, \prec C),\\
&AXA^* \succcurlyeq B\,(\succ B, \preccurlyeq B, \prec B),\\
&AX + (AX)^* \succcurlyeq B\,(\succ B, \preccurlyeq B, \prec B)
\end{aligned}
$$

in the Löwner partial ordering by using ranks, inertias and generalized inverses of matrices.

Mathematics subject classification (2010): Primary 15A24; Secondary 15A09, 15A39, 15A45, 15B57.
Keywords and phrases: Matrix equation, linear matrix inequality, Löwner partial ordering, general solution, generalized inverse of matrix, rank, inertia, relaxation method.

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