

GENERALIZED FRACTIONAL MAXIMAL FUNCTIONS IN LORENTZ SPACES Λ

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Abstract. In this paper we give the complete characterization of the boundedness of generalized fractional maximal operator

$$M_{\phi, \Lambda^\alpha} f(x) := \sup_{Q \ni x} \frac{\|f \chi_Q\|_{\Lambda^\alpha(b)}}{\phi(|Q|)} \quad (x \in \mathbb{R}^n),$$

between the classical Lorentz spaces $\Lambda^p(v)$ and $\Lambda^q(w)$, as well as between $\Lambda^p(v)$ and weak-type Lorentz spaces $\Lambda^{q, \infty}(w)$, and between $\Lambda^{p, \infty}(v)$ and $\Lambda^{q, \infty}(w)$, and between $\Lambda^{p, \infty}(v)$ and $\Lambda^q(w)$, for appropriate functions ϕ , where $0 < p, q, \alpha < \infty$, v, w, b are weights on $(0, \infty)$ such that $0 < B(t) := \int_0^t b < \infty$, $t > 0$, $B \in \Delta_2$ and $B(t)/t^r$ is quasi-increasing for some $0 < r \leq 1$.

Mathematics subject classification (2010): 42B25, 42B35.

Keywords and phrases: Maximal functions, classical and weak-type Lorentz spaces, iterated Hardy inequalities involving suprema, weights.

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