

THE SZEGŐ–MARKOV–BERNSTEIN INEQUALITIES AND BARYCENTRIC REPRESENTATIONS OF THE OSCULATORY INTERPOLATING OPERATORS FOR CLASSICAL ITERATED WEIGHTS

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Abstract. We study inequalities of Szegő–Markov–Bernstein types, barycentric representations of the Lagrange, Fejér and Hermite interpolating operators, and the Gauss quadrature formulae for all iterated weights $w_k(x) = A^k(x)w(x)$ of classical weight functions $w(x)$. In particular, we establish the explicit formulae for the best constants, extremal polynomials and Christoffel numbers, associated with the iterated weight functions of six basic classical weights of Hermite, Laguerre, Jacobi, generalized Bessel, Jacobi on $(0, +\infty)$ and pseudo-Jacobi kind. It should be noted that the results on Markov–Bernstein inequalities continue the investigations of the best constants and extremal polynomials by Guessab and Milovanović [J. Math. Anal. Appl. 182 (1994), pp. 244–249] and Agarwal and Milovanović [Appl. Math. Comput. 128 (2002), pp. 151–166], without any additional assumptions on classical weight functions. Moreover, the presented generic formulae for the Christoffel numbers of the iterated Gauss quadrature rules, together with the corresponding representations of the barycentric weights of Lagrange, Fejér and Hermite type, complete the recent results of Wang et al. and the authors, published in [Math. Comp. 81 (2012) and 83 (2014), pp. 861–877 and 2893–2914, respectively] and [Math. Comp. 86 (2017), pp. 2409–2427].

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