

SHARP BOUNDS FOR SÁNDOR–YANG MEANS IN TERMS OF ONE-PARAMETER FAMILY OF BIVARIATE MEANS

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Abstract. In the article, we present the best possible parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3$ and β_4 on the interval $(0, 1)$ such that the double inequalities

$$G_{\alpha_1}(x, y) < R_{GQ}(x, y) < G_{\beta_1}(x, y), \quad Q_{\alpha_2}(x, y) < R_{QG}(x, y) < Q_{\beta_2}(x, y),$$

$$H_{\alpha_3}(x, y) < R_{GQ}(x, y) < H_{\beta_3}(x, y), \quad C_{\alpha_4}(x, y) < R_{QG}(x, y) < C_{\beta_4}(x, y)$$

hold for all $x, y > 0$ with $x \neq y$, where $R_{GQ}(x, y)$ and $R_{QG}(x, y)$ are the Sándor–Yang means, $H_p(x, y)$, $G_p(x, y)$, $Q_p(x, y)$ and $C_p(x, y)$ are the one-parameter means.

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