

SHARPNESS AND GENERALIZATION OF JORDAN, BECKER–STARK AND PAPENFUSS INEQUALITIES WITH AN APPLICATION

BO ZHANG AND CHAO-PING CHEN

Abstract. In this paper, we present an identity related to Jordan's inequality. More precisely, we provide a formula for determining the coefficients $b_n \equiv b_n(\theta)$ such that

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} b_n (\pi^\theta - (2x)^\theta)^n,$$

where $\theta \geq 2$ is a given real number. We present a generalization of Jordan's inequality. As an application, we improve the well-known Yang Le inequality. We establish sharp bounds for $(\tan x/x)^{(n)}$ for $n = 0$ and $n = 1$. Further, an interesting open problem and a conjecture regarding our present concern are posed.

Mathematics subject classification (2010): 26D05, 41A10.

Keywords and phrases: Jordan's inequality; Becker-Stark inequality; Papenfuss inequality; Yang Le inequality; Bell polynomials of the second kind; Faà di Bruno formula.

REFERENCES

- [1] M. ABRAMOWITZ AND I.A. STEGUN (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Dover, New York, 1965.
- [2] G. D. ANDERSON, S.-L. QIU, M. K. VAMANAMURTHY AND M. VUORINEN, *Generalized elliptic integral and modular equations*, *Pacific J. Math.* **192** (2000), 1–37.
- [3] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, New York, 1997.
- [4] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Monotonicity of Some Functions in Calculus*; Available at <http://www.math.auckland.ac.nz/Research/Reports/Series/538.pdf>.
- [5] G. BACH, *Trigonometric inequality*, *Amer. Math. Monthly* **87** (1) (1980), 62.
- [6] B. BANJAC, M. MAKRAČIĆ AND B. MALEŠEVIĆ, *Some notes on a method for proving inequalities by computer*, *Results. Math.* **69** (2016), 161–176.
- [7] M. BECKER AND E. L. STARK, *On a hierarchy of quolynomial inequalities for $\tan x$* , *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No.* **602-633** (1978), 133–138.
- [8] C.-P. CHEN, *Complete monotonicity and logarithmically complete monotonicity properties for the gamma and psi functions*, *J. Math. Anal. Appl.* **336** (2007), 812–822.
- [9] C.-P. CHEN AND W.-S. CHEUNG, *Sharp Cusa and Becker–Stark inequalities*, *J. Inequal. Appl.* (2011) 136, <http://www.journalofinequalitiesandapplications.com/content/2011/1/136>.
- [10] C.-P. CHEN AND L. DEBNATH, *Sharpness and Generalization of Jordan's Inequality and its applications*, *Appl. Math. Lett.* **25** (2012), 594–599.
- [11] C.-P. CHEN AND N. ELEZOVIĆ, *Sharp Redheffer-type and Becker-Stark-type inequalities with an application*, *Math. Inequal. Appl.* **21** (4) (2018), 1059–1078.
- [12] C.-P. CHEN AND R.B. PARIS, *Series representations of the remainders in the expansions for certain trigonometric functions and some related inequalities, I*, *Math. Inequal. Appl.* **20** (2017), 1003–1016.
- [13] C.-P. CHEN AND R.B. PARIS, *Series representations of the remainders in the expansions for certain trigonometric functions and some related inequalities, II*, *RGMI Res. Rep. Coll.* **20** (2017), Art. 152, 17 pp. <http://rgmia.org/v20.php>.

- [14] L. COMTET, *Advanced Combinatorics*, D. Reidel Publishing Co., Dordrecht, 1974.
- [15] D. CVIJOVIĆ, *New identities for the partial Bell polynomials*, Appl. Math. Lett. **24** (2011), 1544–1547.
- [16] L. DEBNATH, C. MORTICI AND L. ZHU, *Refinements of Jordan–Stečkin and Becker–Stark inequalities*, Results Math. **67** (2015), 207–215.
- [17] L. DEBNATH AND C. J. ZHAO, *New strengthened Jordan’s inequality and its applications*, Appl. Math. Lett. **16** (2003), 557–560.
- [18] K. DENG, *On extensions of the Jordan’s inequality*, J. Xiangtan Mining Inst. **10** (1995), 60–63 (in Chinese).
- [19] H.-F. GE, *New Sharp Bounds for the Bernoulli Numbers and Refinement of Becker–Stark Inequalities*, J. Appl. Math. (2012) Article ID 137507, 7 pages.
<https://www.hindawi.com/journals/jam/2012/137507/>
- [20] W. D. JIANG AND H. YUN, *Sharpening of Jordan’s inequality and its applications*, J. Inequal. Pure Appl. Math. **7** (3) (2006), Article 102.
- [21] J. L. LI, *An identity related to Jordan’s inequality*, Internat. J. Math. Math. Sci. (2006), Article ID 76782.
- [22] B.J. MALEŠEVIĆ, *One method for proving inequalities by computer*, J. Inequal. Appl. (2007) Article ID 78691.
- [23] A. MCD. MERCER, U. ABEL AND D. CACCIA, *A sharpening of Jordan’s inequality*, Amer. Math. Monthly **93** (1986), 568–569.
- [24] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer, New York, 1970.
- [25] Y. NISHIZAWA, *Sharpening of Jordan’s type and Shafer–Fink’s type inequalities with exponential approximations*, Appl. Math. Comput. **269** (2015), 146–154.
- [26] Y. NISHIZAWA, *Sharp Becker–Stark’s type inequalities with power exponential functions*, J. Inequal. Appl. (2015) 402,
<http://www.journalofinequalitiesandapplications.com/content/2015/1/402>.
- [27] D.-W. NIU, Z.-H. HUO, J. CAO AND F. QI, *A general refinement of Jordan’s inequality and a refinement of L. Yang’s inequality*, Integral Transforms Spec. Funct. **19** (2008), 157–164.
- [28] A. Y. ÖZBAN, *A new refined form of Jordan’s inequality and its applications*, Appl. Math. Lett. **19** (2006), 155–160.
- [29] M. C. PAPPENFUSS, *Problem E2739*, Amer. Math. Monthly **85** (9) (1978), 765.
- [30] F. QI, *Extension and sharpenings of Jordan’s and Kober’s inequalities*, Journal of Mathematics for Technology **4** (1996), 98–101 (in Chinese).
- [31] S. B. STEČKIN, *Some remarks on trigonometric polynomials*, Uspekhi Matematicheskikh Nauk, vol. 10, no. 1 (63), (1955) 159–166 (in Russian).
- [32] Z. SUN AND L. ZHU, *Some refinements of inequalities for circular functions*, J. Appl. Math. (2011) Article ID 869261, <http://www.hindawi.com/journals/jam/2011/869261/>.
- [33] Z.-J. SUN AND L. ZHU, *Simple proofs of the Cusa–Huygens–type and Becker–Stark–type inequalities*, J. Math. Inequal. **7** (2013), 563–567.
- [34] S. H. WU, *On generalizations and refinements of Jordan type inequality*, Octog. Math. Mag. **12** (2004), 267–272.
- [35] S. H. WU, *Sharpness and generalization of Jordan’s inequality and its application*, Taiwanese J. Math. **12** (2008), 325–336.
- [36] S. H. WU AND L. DEBNATH, *A new generalized and sharp version of Jordan’s inequality and its applications to the improvement of the Yang–Le inequality*, Appl. Math. Lett. **19** (2006), 1378–1384.
- [37] S. H. WU AND L. DEBNATH, *A new generalized and sharp version of Jordan’s inequality and its applications to the improvement of the Yang–Le inequality, II*, Appl. Math. Lett. **20** (2007), 532–538.
- [38] S. H. WU AND L. DEBNATH, *Jordan-type inequalities for differentiable functions and their applications*, Appl. Math. Lett. **21** (2008), 803–809.
- [39] S. H. WU AND L. DEBNATH, *Generalizations of a parameterized Jordan-type inequality, Janous’s inequality and Tsintsifas’s inequality*, Appl. Math. Lett. **22** (2009), 130–135.
- [40] S. H. WU AND H. M. SRIVASTAVA, *A further refinement of a Jordan type inequality and its application*, Appl. Math. Comput. **197** (2008), 914–923.
- [41] L. YANG, *Distribution of Values and New Research*, Science Press, Beijing, 1982 (in Chinese).
- [42] F. YUEFENG, *Jordan’s inequality*, Math. Mag. **69** (1996), 126–127.
- [43] X. H. ZHANG, G. D. WANG AND Y. M. CHU, *Extensions and sharpenings of Jordan’s and Kober’s inequalities*, J. Inequal. Pure Appl. Math. **7** (2006) no. 2, Article 63.

- [44] C. J. ZHAO, *Generalization and strengthen of Yang Le inequality*, Mathematics Practice Theory **30** (2000), 493–497 (in Chinese).
- [45] C. J. ZHAO AND L. DEBNATH, *On generalizations of L. Yang's inequality*, J. Inequal. Pure Appl. Math. **3** (4) (2002), Article 56.
- [46] L. ZHU, *Sharpening of Jordan's inequalities and its applications*, Math. Inequal. Appl. **9** (2006), 103–106.
- [47] L. ZHU, *Sharpening Jordan's inequality and Yang Le inequality*, Appl. Math. Lett. **19** (2006), 240–243.
- [48] L. ZHU, *Sharpening Jordan's inequality and Yang Le inequality, II*, Appl. Math. Lett. **19** (2006), 990–994.
- [49] L. ZHU, *A general refinement of Jordan-type inequality*, Comput. Math. Appl. **55** (2008), 2498–2505.
- [50] L. ZHU, *General forms of Jordan and Yang Le inequalities*, Appl. Math. Lett. **22** (2009), 236–241.
- [51] L. ZHU, *An extended Jordan's inequality in exponential type*, Appl. Math. Lett. **24** (2011), 1870–1873.
- [52] L. ZHU, *Sharp Becker-Stark-type inequalities for Bessel functions*, J. Inequal. Appl. 2010, Article ID 838740, <http://www.journalofinequalitiesandapplications.com/content/2010/1/838740>.
- [53] L. ZHU, *A refinement of the Becker-Stark inequalities*, Math. Notes **93** (3–4) (2013), 421–425.
- [54] L. ZHU AND J.K. HUA, *Sharpening the Becker-Stark inequalities*, J. Inequal. Appl. 2010, Article ID 931275, <http://www.journalofinequalitiesandapplications.com/content/2010/1/931275>.