

## A NECESSARY AND SUFFICIENT CONDITION FOR THE CONVEXITY OF THE GENERALIZED ELLIPTIC INTEGRAL OF THE FIRST KIND WITH RESPECT TO HÖLDER MEAN

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**Abstract.** In the article, we present the necessary and sufficient conditions such that the generalized elliptic integral and complete  $p$ -elliptic integral of the first kind are convex with respect to Hölder mean, which generalize and refine some previously known results.

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