

## GENERALIZATIONS AND SHARPENINGS OF CERTAIN BERNSTEIN AND TURÁN TYPES OF INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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*Abstract.* Let  $p(z)$  be a polynomial of degree  $n$ . The polar derivative of  $p(z)$  with respect to a complex number  $\alpha$  is defined by

$$D_{\alpha}p(z) = np(z) + (\alpha - z)p'(z).$$

If  $p(z)$  has all its zeros in  $|z| \leq k$ ,  $k \geq 1$ , then for  $|\alpha| \geq k$ , Aziz and Rather [Math. Inequal. Appl., 1, (1998), 231–238] proved

$$\max_{|z|=1} |D_{\alpha}p(z)| \geq n \left( \frac{|\alpha| - k}{1 + k^n} \right) \max_{|z|=1} |p(z)|.$$

In this paper, we first improve as well as generalize the above inequality. Besides, we are able to prove an improvement of a result due to Govil and Mctume [Acta Math. Hungar., 104, (2004), 115–126] and also prove an inequality for a subclass of polynomials having all its zeros in  $|z| \geq k$ ,  $k \leq 1$ .

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