

## CHARACTERIZATIONS FOR THE FRACTIONAL INTEGRAL OPERATOR AND ITS COMMUTATORS IN GENERALIZED WEIGHTED MORREY SPACES ON CARNOT GROUPS

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**Abstract.** In this paper, we shall give a characterization for the strong and weak type Spanne type boundedness of the fractional integral operator  $I_\alpha$ ,  $0 < \alpha < Q$  on Carnot group  $\mathbb{G}$  on generalized weighted Morrey spaces  $M_{p,\varphi}(\mathbb{G}, w)$ , respectively, where  $Q$  is the homogeneous dimension of  $\mathbb{G}$ . Also we give a characterization for the Spanne type boundedness of the commutator operator  $[b, I_\alpha]$  on generalized weighted Morrey spaces.

As applications of the properties of the fundamental solution of sub-Laplacian  $\mathcal{L}$  on  $\mathbb{G}$ , we prove two Sobolev-Stein embedding theorems on generalized weighted Morrey spaces in the Carnot group setting.

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