

ON POST QUANTUM INTEGRAL INEQUALITIES

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Abstract. In the article, we provide some new post quantum refinements of the Hermite-Hadamard like inequalities involving the class of h -preinvex functions by establishing a new auxiliary result involving the post quantum differentiable function. By discussing some special cases, it is shown that our obtained results are the further generalizations of many previous known results.

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