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A GENERAL NONLINEAR VERSION OF ROTH'S THEOREM ON THE REAL LINE

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Abstract. Let $N > 1$ be a real number and $\varepsilon > 0$ be given. In this paper, we will prove that, for a measurable subset S of $[0,N]$ with positive density ε, there must be patterns of the form $(x, x+t, x+\gamma(t))$ such that

$$x, x+t, x+\gamma(t) \in S,$$

where γ is convex and has some curvature constraints, $t > \delta(\varepsilon,\gamma)\gamma^{-1}(N)$ and $\delta(\varepsilon,\gamma)$ is a positive constant depending only on ε and γ, γ^{-1} is the inverse function of γ. Our result extends Bourgain's result [2] to the general curve γ. We use Bourgain's energy pigeonholing argument and Li's σ-uniformity argument.

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