

THE NONEXISTENCE OF EXTREMALS FOR THE HARDY–TRUDINGER–MOSER INEQUALITY IN THE HYPERBOLIC SPACE

QIANJIN LUO

Abstract. Let \mathbb{B} be the unit disc in \mathbb{R}^2 , \mathcal{H} be the completion of $C_0^\infty(\mathbb{B})$ under the norm

$$\|u\|_{\mathcal{H}} = \left(\int_{\mathbb{B}} |\nabla u|^2 dx - \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} dx \right)^{\frac{1}{2}}, \quad \forall u \in C_0^\infty(\mathbb{B}).$$

We prove that the supremum in the following inequality

$$\sup_{u \in \mathcal{H}, \|u\|_{\mathcal{H}} \leq 1} \int_{\mathbb{B}} \exp\{4\pi(1 + \alpha\|u\|_2^2)u^2\} dx < +\infty$$

can not be achieved by any functions in the function space \mathcal{H} when α is sufficiently close to λ_1^- , i.e., $0 < \lambda_1 - \alpha \ll 1$, where

$$\lambda_1(\mathbb{B}) = \inf_{u \in \mathcal{H}, u \neq 0} \frac{\|u\|_{\mathcal{H}}^2}{\|u\|_2^2}.$$

Evidently, this conclusion is complementary to that of [12, Theorem 1.1 (ii)].

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