

## APPLICATIONS OF A DUALITY BETWEEN GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS II

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**Abstract.** Generalized trigonometric functions and generalized hyperbolic functions can be converted to each other by the duality formulas previously discovered by the authors. In this paper, we apply the duality formulas to prove dual pairs of Wilker-type inequalities, Huygens-type inequalities, and (relaxed) Cusa-Huygens-type inequalities for the generalized functions. In addition, multiple- and double-angle formulas not previously obtained are also given.

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