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OPTIMAL LEHMER MEAN BOUNDS FOR THE nTH POWER–TYPE TOADER MEANS OF $n=-1,1,3$

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Abstract. In the article, we prove that $\lambda_1=0$, $\mu_1=5/8$, $\lambda_2=-1/8$, $\mu_2=0$, $\lambda_3=-1$ and $\mu_3=-7/8$ are the best possible parameters such that the double inequalities

$$L_{\lambda_1}(a,b)<T_3(a,b)<L_{\mu_1}(a,b),$$
$$L_{\lambda_2}(a,b)<T_1(a,b)<L_{\mu_2}(a,b),$$
$$L_{\lambda_3}(a,b)<T_{-1}(a,b)<L_{\mu_3}(a,b)$$

hold for $a,b>0$ with $a\neq b$, and provide new bounds for the complete elliptic integral of the second kind $\mathscr{E}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta$ on the interval $(0,1)$, where $L_p(a,b)=(a^{p+1}+b^{p+1})/(a^p+b^p)$ is the p-th Lehmer mean and $T_n(a,b)=\left(\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^n\cos^2\theta+b^n\sin^2\theta}d\theta\right)^{2/n}$ is the nth power-type Toader mean.

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