

## ON MULTI-INDEX WHITTAKER FUNCTION, RELATED INTEGRALS AND INEQUALITIES

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*Abstract.* A new generalization of Whittaker function  $M_{\lambda, \mu}(z)$  is introduced and studied by means of the extended multi-index confluent hypergeometric function of the first kind  $\Phi_{(\alpha, \beta)}^{(\gamma); P}$  introduced in [1]. The related Euler-type integral representation and the Laplace–Mellin and Hankel integral transforms are also presented. Functional two-sided bounding inequality is established for the multi-index Mittag-Leffler function, and in continuation functional lower bound is derived for the associated ML-extended Whittaker function.

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